Quantum discord in a Majorana-based device as a characterisation of non-local correlations

PHYS451: MPhys Project

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Abstract

Quantum discord is a measure of nonclassical correlations between subsystems of a quantum system which are not necessarily entangled. We consider a minimal complexity setup consisting of six fermions. In the presence of a superconductor, six Majorana zero modes (MZM) emerge and the system can be partitioned to perform non-trivial measurements on its subsystems. We compute the quantum discord for a class of states (Werner states) within the degenerate ground state. We show that a non-vanishing discord is present for the fully mixed state and we discuss it as a measure of the intrinsic correlation of the system.

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Chapter 1

Introduction

Majorana zero modes (MZM) –a type of localised quasiparticle –hold great promise for topological quantum computing [1]. Over the years they have evolved from a largely theoretical topic into an active experimental field at the forefront of condensed matter physics [2]. This transformation was driven in part by the translation of abstract models [1] into realistic blueprints realisable in the laboratory [3]. Majorana quasiparticles are a twist on an idea of Ettore Majorana. In 1937, Majorana rewrote Dirac's theory of spin–1/2 particles to allow for real wave solutions [4, 5]. The consequence was that Majorana particles are identical to their antiparticles. Indirect signaures of Majorana's particles with a twist are being shown to emerge in one-dimensional wires in the presence of a superconductor and small magnetic field [6, 7, 8]. Their non-Abelian statistics make MZM a candidate for fault-tolerant information processing [9]. Methods for preparation, manipulation, and readout of MZM are being implemented [2]. An important step towards their application for quantum information processing is a thorough understanding of their quantum states. Romito and Gefen [10] have proposed a device in which MZM emerging from topology exhibit ubiquitous non-local quantum entanglement. Interestingly, there may exist non-local quantum correlations which are not due to entanglement.

One measure of quantum correlations is the quantum discord. Quantum correlations were originally thought to be entirely due to entanglement pairs [11]. Ollivier and Zurek [12] and Henderson and Vedral [13] found that it is not necessarily the case. They introduce quantum discord as a measure of genuine quantum correlations which can be present in non-pure non-entangled states. Quantum discord is defined as the difference between two measures of mutual information. One measure of the total information content of a bipartite system is the information of its subsystems as well as their mutual information. In the classical limit, Bayes' Theorem gives an equivalent measure of the total information. For quantum systems the difference between these measures is nonzero due to non-local effects of measurements of noncommutative observables. Quantum discord has been shown to be more robust than entanglement against decoherence in certain enviornments [14, 15] and shows quantum advantage in some computational models without or with little entanglement [16, 17].

Since quantum discord measures quantum correlations not necessarily due to entanglement, it is interesting to see how it behaves in systems with non-trivial non-local correlations originating from topology. Romito and Gefen [10] have devised a minimal complexity setup consisting of six superconducting wires which host Majorana zero modes at their ends. At low energy, the system consists of six Majorana degrees of freedom. The system is partitioned in two and each set probed by an external detector using available techniques. Any allowed state in the degenerate Majorana space is nonlocally entangled. Computing the quantum discord require a full model describing the bulk, not only the Majorana zero energy end states. We devise a minimal complexity device consisting of six fermions which interact via nearest-neighbour hopping and Cooper pairing (i.e. superconductivity). At zero energy and in the limit that the superconducting coherence length is zero, the device hosts six MZM at its edges. They can be probed by the detector devised by Romito and Gefen. Remarkably, the four-degenerate Majorana ground space exhibits similar algebraic structure as two-qubit systems [18]. By showing that the spin–1/2 algebra holds in a simple model and a more general one, we suggest that the algebra is independent of the particular configuration of the junction.

From the four ground states, we can construct a maximally entangled state and the maximally mixed state –a state which is a statistical ensemble of the four pure states. We define the Werner state in the ground space as a linear combination of a maximally entangled and the maximally mixed state. The Werner state traces out a path from the maximally mixed state to the particular maximally entangled state. In two-qubit systems the maximally mixed state generates zero discord. Remarkably, the maximally mixed state in the Majorana setup shows nonzero discord. We show analytically that quantum discord is ubiquitous for any Werner state. We also show that it is monotonically increasing as it traces out its path away from the maximally mixed state. Finally, we show that the Majorana setup exhibit strikingly similar discord to two-qubit systems as a result of the spin–1/2 algebra.

To introduce the reader to quantum discord, we present a classical method for quantifying information (Sec. 2.1.1), known as Shannon entropy. We also introduce a precise reformulation of quantum mechanics (Sec. 2.1.2), known as the von Neumann formulation, in which states are represented by density operators. It is often omitted from introductory quantum mechanics courses in favour of more pragmatic approaches [19]. The density operator formalism allows us to 'trace over' degrees of freedom of a system, enabling us to partition the system into sets. We can then determine the mutual information between these sets, as measured by the quantum mutual information and the classical mutual information.

In the classical limit, the quantum mutual information and the classical mutual information are equivalent. However, for quantum states there is a discrepency between them. This discrepency defines the quantum discord. Few analytical expressions of quantum discord exist. Luo [18] has ingeniously obtained one such solution for two-qubit systems. We review Luo's calculation in section 2.1.6. In order to compare the quantum discord with entanglement, we will need to compare it with a good measure of entanglement. The entanglement entropy is the canonical measure of entanglement for pure states. But for mixed states it is no longer a good measure. Accordingly, we present a more general measure of entanglement in section 2.2, the entanglement of formation. These tools we will need in the investigation of quantum discord in a minimal complexity Majorana-based device, which we model in chapter 3. First we study the entanglement entropy across simple systems (Kitaev chains). Understanding the behaviour of entanglement in these simple systems furthers our analysis of the minimal complexity device. Finally, we obtain an analytical expression of the quantum discord of Werner states in the ground-state space.

Chapter 2

Background

2.1 Quantum discord

Quantum discord is a measure of nonclassical correlations between two subsystems of a quantum system. It was proposed simultaneously by Harold Ollivier and Wojciech H. Zurek [12], and L. Henderson and Vlatko Vedral [13] (See [11] for historical notes.) Two classically identical expressions for the mutual information generally differ when the systems involved are quantum. This difference defines the quantum discord. Separability of the density matrix describing a pair of systems does not guarantee vanishing of the discord, thus showing that absence of entanglement does not imply classicality.

First we introduce a measure of information, known as Shannon entropy. Then we present two classically equivalent ways of quantifying the mutual information of two subsystems using Bayes' Theorem. In a bipartite quantum system, these two quantities are not equivalent due to correlations and noncommutativity of operators. This discrepancy is known as *quantum discord*.

2.1.1 Classical information

Following Quantum Computation and Quantum Information by Nielsen and Chuang [20], we review a measure of uncertainty associated with classical probability distributions, known as the *Shannon entropy*. The Shannon entropy of a discrete random variable X with possible values $\{x_1, \ldots, x_n\}$ and associated probability distribution, $p_1, \ldots, p_n$, is defined as

$$S(X) \equiv S(p_1, \ldots, p_n) \equiv -\sum_x^n p_x \log p_x. \tag{2.1}$$

It quantifies the information gained after a measurement of X or equivalently, our uncertainty about X before the measurement. One can choose to measure information to any base. We choose the minimal one –base two –and refer to the units as 'bits'. The Shannon entropy is the minimum number of bits required to store the information being produced by the source, in such a way that at a later time the information can be reconstructed –the result of *Shannons noiseless coding theorem* [21].

Given two random variables X and Y, the *joint* Shannon entropy is defined as

$$S(X, Y) \equiv -\sum_x \sum_y p(x, y) \log p(x, y), \tag{2.2}$$

where $p(x, y)$ is a joint, or multivariate probability distribution of X and Y.

The joint entropy measures our total uncertainty about the pair of distributions (X, Y). Suppose we know the value of Y, so we have acquired $S(Y)$ bits of information about the pair (X, Y). The remaining uncertainty about the pair (X, Y) is associated with our

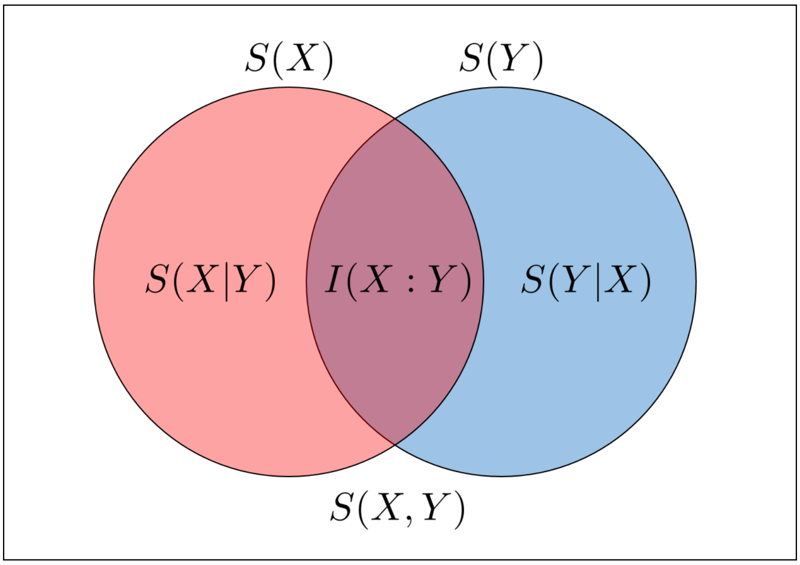


Figure 2.1: Modified from Fig 1.1 of V. Vedral [22]. The relationship between the entropy $S(X)$, the joint entropy $S(X,Y)$, the conditional entropy $S(X|Y)$ and the mutual information $I(X:Y)$. From this diagram, we can see how the different entropies are related by addition; for example, $S(X,Y) = S(X|Y) + I(X:Y) + S(Y|X)$.

remaining lack of knowledge about X, even given that we know Y. The *entropy of X conditional on knowing Y* is therefore defined by

$$S(X|Y) \equiv S(X,Y) - S(Y). \tag{2.3}$$

The *mutual information content of X and Y* measures how much information X and Y have in common

$$I(X:Y) \equiv S(X) + S(Y) - S(X,Y). \tag{2.4}$$

Bayes' Theorem tells us an equivalent way of writing the mutual information is

$$I(X:Y) \equiv S(X) - S(X|Y). \tag{2.5}$$

2.1.2 Von Neumann formalism of quantum mechanics

We now generalise the definition of the Shannon entropy to quantum states following von Neumann [23]. Suppose a system is in one of a number of states $|\psi_i\rangle$, with respective probabilities p_i. Von Neumann defines the *density operator* as

$$\rho \equiv \sum_i p_i |\psi_i\rangle\langle\psi_i|. \tag{2.6}$$

Since the wavefunction is normalised and its associated probabilities are nonzero, ρ is a density operator if and only if its trace is 1 and it is a positive operator.

In the von Neumann formalism, quantum measurements are described by a collection $\{M_m\}$ of *measurement operators*, known as a positive operator valued measurement (POVM) [24]. $\{M_m\}$ are operators acting on the state space of the system being measured. The index m refers to the measurement outcomes that may occur in an experiment. If the state of the quantum system is ρ immediately before the measurement, then the probability that result m occurs is given by

$$p(m) = \mathrm{Tr}(M_m^\dagger M_m \rho), \tag{2.7}$$

and the state of the system after the measurement is

$$\frac{M_m^\dagger \rho M_m}{p(m)}. \tag{2.8}$$

The measurement operators satisfy the *completeness condition*,

$$\sum_m M_m^\dagger M_m = I. \tag{2.9}$$

Finally, the state space of a composite physical system is the tensor product of the state spaces of the component physical systems. Moreover, if we have subsystems numbered 1 through n, and the subsystem number i is prepared in the state ρ_i, then the joint state for the composite system is $\rho_1 \otimes \cdots \otimes \rho_n$.

The density operator allows us to write any state $|\psi\rangle$ as $\rho = |\psi\rangle\langle\psi|$. In this case $\mathrm{Tr}(\rho^2) = 1$ and we say that ρ corresponds to a *pure state*. We may introduce uncertainty into the preparation by writing $\rho = \sum_i p_i |\psi_i\rangle\langle\psi_i|$. In this case ρ may have been prepared in any one of $|\psi_i\rangle$ with corresponding probabilities p_i. Such a ρ satisfies $\mathrm{Tr}(\rho^2) < 1$ and is known as a *mixed state*.

2.1.3 Entanglement entropy

The *von Neumann entropy*, or *entanglement entropy* is a quantum extension of the Shannon entropy, replacing classical probability distributions with density operators acting on a composite Hilbert space and the summation is replaced by the trace

$$S(\rho) \equiv -\operatorname{Tr} \rho \log_2 \rho. \tag{2.10}$$

The spectral theorem allows us to write $\rho = U\Sigma U^\dagger$, where $\Sigma =$Diag(λ_i), λ_i are the eigenvalue of ρ and U is a unitary matrix. This gives us a computationally easier form of the entropy

$$S(\rho) = -\operatorname{Tr}(U\Sigma U^\dagger \log_2(U\Sigma U^\dagger)) = -\operatorname{Tr}(\Sigma \log_2 \Sigma) = -\sum_i \lambda_i \log_2 \lambda_i. \tag{2.11}$$

This affords a general property of the entropy. Given an ensemble $\{|\psi_i\rangle\}_{0<i\leqslant n}$ of orthogonal states, ρ will have an n-dimensional column space $\{|\psi_i\rangle\}_{0<i\leqslant n}$ and $\mathrm{rank}(\rho) = n$. Since $\mathrm{Tr}(\rho) = 1$, pure states will have a single eigenvalue 1 and entropy $S(\rho) = 0$ by l'Hôpital's Rule. Mixed states will have several eigenvalues and $S(\rho) \leq 1$. A maximally mixed state in a Hilbert space of dimension η will have rank η, for example, the identity matrix I/η. Maximally mixed states I/η have entropy $-\sum_i \frac{1}{\eta} \log_2 \frac{1}{\eta} = \log_2 \eta$.

2.1.4 The reduced density operator

Perhaps the deepest application of the density operator is as a virtually indispensable tool in the analysis in the *subsystems* of a composite quantum system. Such a description is provided by the *reduced density operator*.

Suppose we have physical systems A and B, whose state is described by a density operator ρ^{AB}. The reduced density operator for system A is defined as

$$\rho^A \equiv \mathrm{Tr}_B(\rho^{AB}), \tag{2.12}$$

where Tr_B is a linear map of operators known as the the *partial trace* over system B

$$\mathrm{Tr}_B(|a_1\rangle\langle a_2| \otimes |b_1\rangle\langle b_2|) \equiv |a_1\rangle\langle a_2| \operatorname{Tr}(|b_1\rangle\langle b_2|), \tag{2.13}$$

where $|a_1\rangle, |a_2\rangle$ are any two vectors in the state space of A, and $|b_1\rangle, |b_2\rangle$ are any two vectors in the state space of B.

The *Schmidt decomposition* for bipartite systems [25] allows us to pick a basis $\{|i_A\rangle\}$ for subsystem A and $\{|i_B\rangle\}$ for B such that each state $|i_A\rangle$ will be correlated with a particular state $|i_B\rangle$ of B. That is

$$|\psi\rangle = \sum_i \lambda_i |i_A\rangle \otimes |i_B\rangle. \tag{2.14}$$

This simplifies calculations of the partial trace since

$$\rho_A = \sum_i \lambda_i^2 |i_A\rangle\langle i_A| \operatorname{Tr} |i_B\rangle\langle i_B| \tag{2.15}$$

$$= \sum_i \lambda_i^2 |i_A\rangle\langle i_A| \tag{2.16}$$

$$\rho_B = \sum_i \lambda_i^2 |i_B\rangle\langle i_B|, \tag{2.17}$$

where $\operatorname{Tr} |i_B\rangle\langle i_B| = 1$ by normality. Hence the eigenvalues of ρ^A and ρ^B are equal and $S(\rho^A) = S(\rho^B)$. The entanglement entropy of either member of the pair are equal, which we expect intuitively.

2.1.5 Quantum information

Harold Ollivier and Wojciech H. Zurek [12], and L. Henderson and Vlatko Vedral [13] generalise the classical mutual information into the quantum scenario and investigate its consequences and implications.

The first natural extension is the *quantum mutual information*

$$I(\rho) \equiv S(\rho^A) + S(\rho^B) - S(\rho) \tag{2.18}$$

$$= S(\rho^A) - S(\rho|\rho^B) \tag{2.19}$$

where $S(\rho|\rho^B) = S(\rho) - S(\rho^B)$. Classically, entropy quantifies the number of microstates available to the system for a specified state. But bipartite quantum states exist for which the entropy of the subsystems is greater than the entropy of the composite system. So the von Neumann conditional entropy $S(\rho|\rho^B)$ can be negative. A negative entropy does not have an obvious physical meaning. We can define the conditional entropy in another way, so that it is always positive. We define the *quantum conditional entropy* as the average entropy of the states of A after a measurement $\{B_k\}$ is performed on B, weighted by the probability outcome p_k

$$S(\rho|\{B_k\}) \equiv \sum_k p_k S(\rho_k). \tag{2.20}$$

The *measurement-induced quantum mutual information* is

$$I(\rho|\{B_k\}) = S(\rho^A) - S(\rho|\{B_k\}). \tag{2.21}$$

There are infinitely many measurements we can perform on B, so we will choose the one that makes $S(\rho|\{B_k\})$ minimum (we want to learn as much about A by measuring B). Then the *classical mutual information* is

$$C(\rho) = S(\rho^A) - \inf_{\{B_k\}} \left(S(\rho|\{B_k\})\right) \tag{2.22}$$

$$= \sup_{\{B_k\}} I(\rho|\{B_k\}). \tag{2.23}$$

Unlike in classical information, the two ways of expressing quantum mutual information are actually *different*. This is because the quantum mutual information can actually reach the value of $2S(\rho^A)$, while the classical mutual information is bounded between 0 and $S(\rho^A)$.

This discrepancy between two natural yet different quantum analogs of the classical mutual information is is known as the *quantum discord*

$$Q(\rho) \equiv I(\rho) - C(\rho) \tag{2.24}$$

and quantifies measurement-induced nonclassical correlations. For entangled pure states the discord is one and for unentangled pure states it's zero. Ollivier and Zurek, and Henderson and Vedral found that for mixed states the discord takes on intermediate values between zero and one. Importantly, discord can exist even we there is no entanglement, as will will see in section 2.2. Quantum discord in the absence of entanglement is due to the noncommutativity of quantum operators.

Discordant states may be useful for quantum information processing [26, 27]. Vedral [11] noted (Feb 2017) that it is still an open question if universal quantum computation can be done without entanglement in the general case of mixed states.

2.1.6 Quantum discord for two-qubit systems

Due to the complicated optimisation invloved, it is usually intractable to evaluate the quantum discord for generic cases. Nevertheless, S. Luo [18] evaluates the discord analytically for a large family of two-*qubit* systems. By "qubits" we mean two-level quantum systems –such as the spin of a spin–1/2 particle or the polarization of a photon.

Two-qubit systems are described by Hilbert space $\mathbb{C}^2 \otimes \mathbb{C}^2$. We choose the standard computational base $\{\,|00\rangle, |01\rangle, |10\rangle, |11\rangle\,\}$. A general two-qubit bipartite state is local unitary equivalent to

$$\rho = \frac{1}{4}\left(I + \sum_{j=1}^{3} c_j \sigma_j \otimes \sigma_j\right), \tag{2.25}$$

where c_j are real constants satisfying certain constraints such that ρ is a well-defined density operator and σ_j are the Pauli spin–1/2 matrices. (By local unitary equivalence, we mean that we can always change the chosen basis for each qubit.) The eigenvalues $\{\lambda_l\}_{l\in\{0,1,2,3\}}$ of ρ depend on c_1, c_2, c_3 such that $\lambda_l \in [0,1]$. The reduced density operators $\rho^a = \rho^b = I/2$. Consequently, the quantum mutual information in ρ is

$$I(\rho) = 2 + \sum_{l=0}^{3} \lambda_l \log_2 \lambda_l. \tag{2.26}$$

Each qubit will be in one of two eigenstates of the basis $\{|k\rangle\}$, and since the measurement can be performed in an arbitrary basis $V \in \mathbf{U}(2)$, the local von Neumann measurement for party b can be written as

$$\{B_k = V\Pi_k V^\dagger : k = 0, 1\}. \tag{2.27}$$

After the measurement $\{B_k\}$, the state ρ will change to the ensemble $\{\rho_k, p_k\}$ with

$$\rho_k := \frac{1}{p_k}(I \otimes B_k)\rho(I \otimes B_k) \tag{2.28}$$

and $p_k := \mathrm{Tr}(I\otimes B_k)\rho(I\otimes B_k)$. For simplicity, let $c := \max\{|c_1|, |c_2|, |c_3|\}$. As expected, the quantum conditional entropy $S(\rho|\{B_k\})$ and the quantum mutual information $I(\rho|\{B_k\})$

are functions of the state ρ and measurement $\{B_k\}$, but the classical mutual information is a function of the state only

$$C(\rho) = \frac{1-c}{2}\log_2(1-c) + \frac{1+c}{2}\log_2(1+c). \tag{2.29}$$

Thus the quantum discord is

$$\begin{aligned} Q(\rho) &= I(\rho) - C(\rho) & (2.30)\\ &= 2 + \sum_{l=0}^{3}\lambda_l \log_2 \lambda_l - \frac{1-c}{2}\log_2(1-c) - \frac{1+c}{2}\log_2(1+c). & (2.31)\end{aligned}$$

In the particular case $c_1 = c_2 = c_3 = -c$, ρ turns out to be the *Werner state*

$$\rho = (1-c)\frac{I}{4} + c|\Psi^-\rangle\langle\Psi^-|, c \in [0,1] \tag{2.32}$$

where $|\Psi^-\rangle = \frac{1}{\sqrt{2}}(|01\rangle - |10\rangle)$ is a maximally entangled Bell state and $I/4$ is a maximally mixed state. Accordingly, we have

$$I(\rho) = \frac{3(1-c)}{4}\log_2(1-c) + \frac{1+3c}{4}\log_2(1+3c), \tag{2.33}$$

$$C(\rho) = \frac{1-c}{2}\log_2(1-c) + \frac{1+c}{2}\log_2(1+c), \tag{2.34}$$

and the quantum discord

$$Q(\rho) = \frac{1-c}{4}\log_2(1-c) - \frac{1+c}{2}\log_2(1+c) + \frac{1+3c}{4}\log_2(1+3c). \tag{2.35}$$

In the case of the maximally entangled Bell state $|\Psi^-\rangle\langle\Psi^-|$, we have that the total correlations, $I(\rho) = 2$ are equally divided into the classical, $C(\rho) = 1$ and quantum, $Q(\rho) = 1$ correlations. The maximally mixed state $I/4$ is essentially an operator formalism of the classical bipartite probability distribution (Eq. 2.4) without any quantum nature, $Q(\rho) = 0$. The von Neumann generalisations of the total and classical mutual information equal the Shannon mutual information and there are no quantum correlations.

It is interesting to compare the quantum discord with the entanglement, and in particular, inquire whether they give the same qualitative characterisations of quantum correlations. Accordingly, we review a measure of entanglement in the following section, and following Luo, compare the discord with the entanglement for two-qubit systems.

2.2 Entanglement of formation

One of the main goals in quantum information theory is to develop a theory of entanglement [28]. Entanglement is a property of bipartite systems –that is systems consisting of two parts A and B that are too far apart to interact, and whose state, lies in a Hilbert space $\mathcal{H}_A \otimes \mathcal{H}_B$. A cornerstone of this theory is a good measure of bipartite entanglement. Measurements are performed by two observers, "Alice" and "Bob" each having access to one of the subsystems. We allow Alice and Bob to perform local operations (LO), e.g., unitary transformations and measurements, on their respective subsystems and to communicate with each other classically (CC). For pure bipartite states a good measure of entanglement has been found: the reduced Von Neumann entropy, defined in section 2.1.4.

The situation for mixed states is much more complex –it can be measured via different quantifiers, which, in general, do not coincide with each other. One of them is *entanglement*

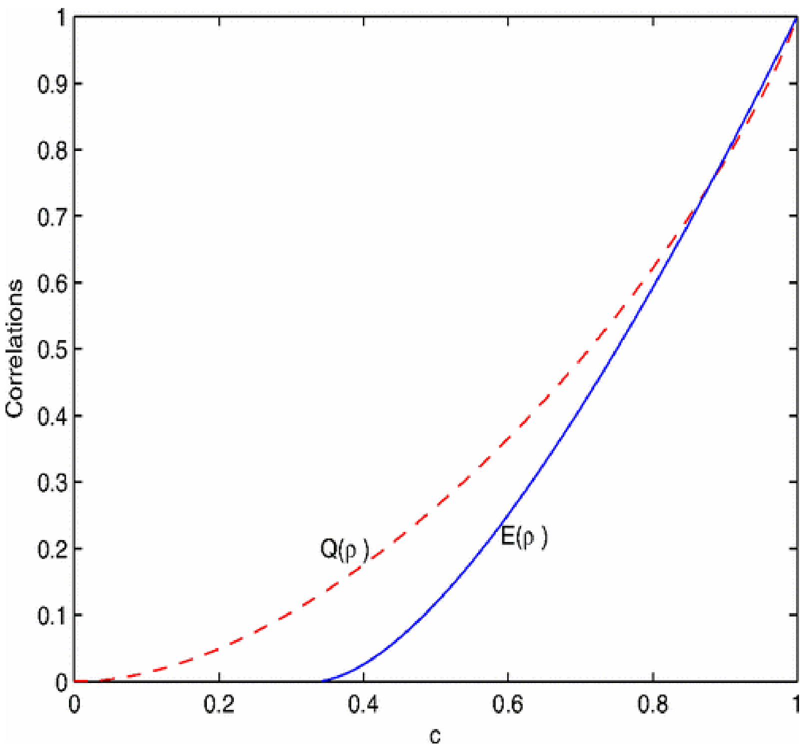


Figure 2.2: From S. Luo [18] Quantum discord $Q(\rho)$ and entanglement of formation $\mathcal{E}(\rho)$ versus c for the Werner state $\rho = (1-c)\frac{I}{4} + c|\Psi^-\rangle\langle\Psi^-|$. Here we see that there are no simple dominance relations. Indeed, $Q(\rho) < \mathcal{E}(\rho)$ when $c \in (0.879, 1)$ and $Q(\rho) > \mathcal{E}(\rho)$, otherwise. The correlations have unit 'bit'.

of formation, introduced by Bennet *et al.* [29]. This measure is intended to quantify the number of singlet states needed to create a given state. It is defined as the minimum average entanglement of an ensemble of pure states that represents the given mixed state $\rho = \sum_i p_i \rho_i$,

$$E(\rho) = \min \sum_i p_i S(\rho_i^A) = \min \sum_i p_i S(\rho_i^B). \tag{2.36}$$

The entanglement of formation involves extremisations that are difficult to handle analytically. However, in the special case of entanglement between two binary quantum systems ("qubits") an explicit formula for arbitrary states has ingeniously been found by Wooters [30, 31]:

In the standard basis $\{|\uparrow\uparrow\rangle, |\uparrow\downarrow\rangle, |\downarrow\uparrow\rangle, |\downarrow\downarrow\rangle\}$, define the "spin flip" transformation on the state of a pair of qubits ρ by

$$\tilde{\rho} = (\sigma_2 \otimes \sigma_2)\rho^*(\sigma_2 \otimes \sigma_2), \tag{2.37}$$

and the *concurrence* as

$$\theta(\rho) = \max\{0, \theta_1 - \theta_2 - \theta_3 - \theta_4\}, \tag{2.38}$$

where the θ_i's are the square roots of the eigenvalues of $\rho\tilde{\rho}$. The formula for the *entanglement of formation* of a mixed state ρ of two qubits is then

$$E(\rho) = \mathcal{E}(\theta(\rho)), \tag{2.39}$$

where the function $\mathcal{E}$ is defined by

$$\mathcal{E}(\theta) = h\left(\frac{1+\sqrt{1-\theta^2}}{2}\right); \tag{2.40}$$

$$h(x) = -x \log_2 x - (1-x) \log_2(1-x). \tag{2.41}$$

S. Luo [18] plots the entanglement of formation against the quantum discord for the Werner state (Fig. 2.2). Interestingly, there are no simple dominance relations between them. They are incomparable in the sense that there exist states in which the discord is larger than the entanglement and conversely, states in which the entanglement is larger than the discord. Discord and entanglement are different not only quantitatively, but also qualitatively.

2.3 Quantum correlations (entanglement) in topologically protected systems

Since quantum discord measures quantum correlations not necessarily due to entanglement, it is interesting to see how it behaves in systems with non-trivial non-local correlations originating from topology. Romito and Gefen [10] have devised a minimal complexity setup consisting of six Majorana zero modes (MZM). They find that any allowed state in the degenerate Majorana space is nonlocally entangled. They show how to measure the presence of this entanglement using available techniques.

The minimal complexity setup consists of a multiterminal junction made up of six one-dimensional topological superconductors (branches) which meet at a common point, depicted in Fig. 2.3. Each branch $\alpha \in \{1, \ldots, 6\}$ is a one-dimensional spinless p-wave superconductor characterised by a gap Δ_α and by MZM $\gamma_\alpha, \gamma'_\alpha$ at the edges of the wire. At the junction, Josephson coupling pairs up the Majorana modes in the junction $\gamma'_1, \ldots, \gamma'_6$ to finite energy states. These states are then gapped out of the ground-state space. The MZM $\gamma_1, \ldots, \gamma_6$ far from the junction are the remaining zero energy degrees of freedom, which span a 2^3 degenerate ground state. The ground space may exchange pairs of fermions with the underlying superconductor, so that the number of fermions is not well-defined. However, the parity of the Majorana system is a good quantum number, so that the degenerate ground space accommodates two subspaces, each of a definite parity. Without loss of generality, we restrict ourselves to the four-dimensional odd subspace.

The MZM $\gamma_1, \ldots, \gamma_6$ can be partitioned into two different sets, left (L) and right (R). We depict the four possible paritionings in Fig. [10]. A separate external detector can be tuned to measure any combination of pairs of products of MZM. We focus on example, Fig. 2.3(a), in which the left set consists of γ_1, γ_3 and γ_5 and the external detector can measure any operator of the form

$$\hat{O}_L = \cos\theta_L \sigma_{z,L} + \sin\theta_L \cos\phi_L \sigma_{x,L} + \sin\theta_L \sin\phi_L \sigma_{y,L}, \tag{2.42}$$

where $\sigma_{z,L} \equiv -i\gamma_1\gamma_3, \sigma_{x,L} \equiv -i\gamma_3\gamma_5$ and $\sigma_{y,L} \equiv -i\gamma_5\gamma_1$. Details of the measurement procedure are discussed by Romito and Gefen.

It turns out that the expectation values of measurements are bounded, $-1 \leq \langle \hat{O}_L \rangle \leq 1$. Genuine quantum correlations underlying a state can be identified through the expectation value of correlated measurements. Bell's theorem asserts that a state within a local hidden variables theory (also known as local realism) satisfies the Bell inequality [32] and (more conducive to experimental testing) the Clauser-Horne-Shimony-Holt (CHSH) [33] inequality

$$\mathcal{C} = |\langle \hat{O}_L \hat{O}_R \rangle - \langle \hat{O}_L \hat{O}'_R \rangle| + |\langle \hat{O}'_L \hat{O}'_R \rangle + \langle \hat{O}'_L \hat{O}_R \rangle| \leq 2, \tag{2.43}$$

where $\hat{O}_L, \hat{O}'_L$ and $\hat{O}_R, \hat{O}'_R$ are pairs of spatially separable sets of observables. Instead, for non-locally entangled quantum states, measurements can satisfy $2 < \mathcal{C} \leq 2\sqrt{2}$ [34, 35, 36, 37], providing evidence of genuine quantum correlations.

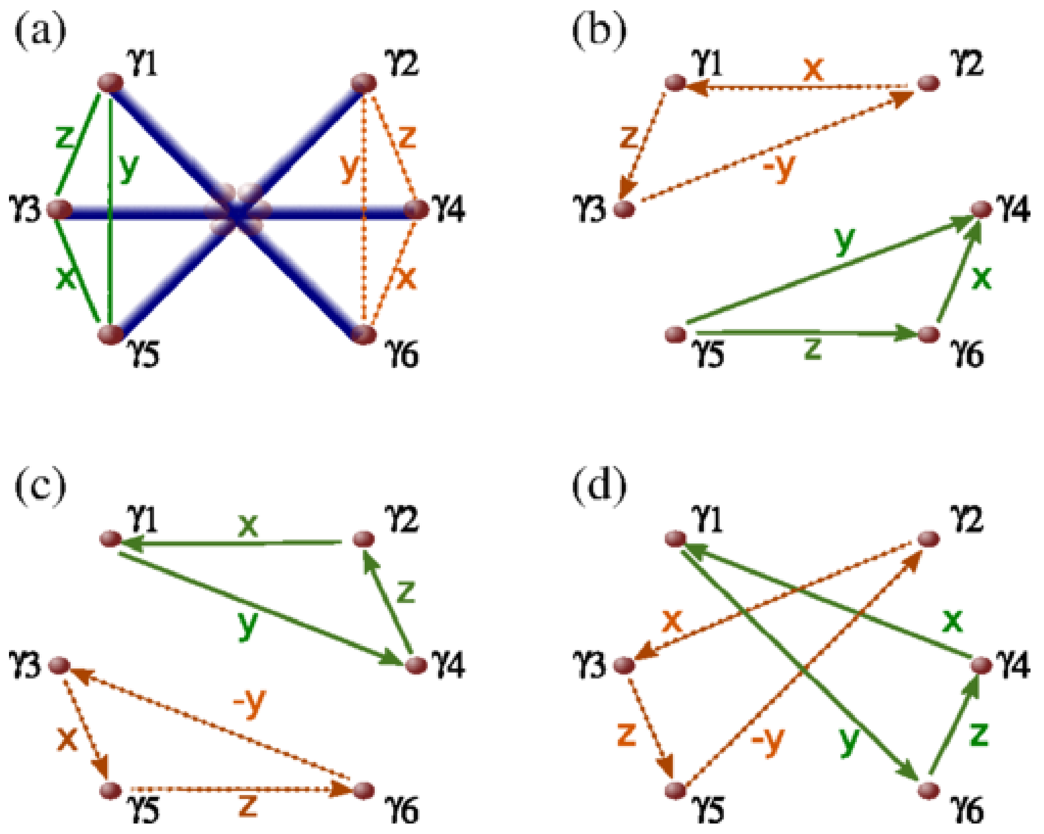


Figure 2.3: From Romito and Gefen [10]. Romito and Gefen's minimal complexity setup is a multiterminal junction consisting of six branches (blue). Branches are topological superconductors which host Majorana zero modes (MZM) at their edges (red dots). MZM at the centre (faded red) couple and are projected out of the ground state-space. The six MZM at the ends (solid red) $\gamma_1, \ldots, \gamma_6$ constitute the zero energy degrees of freedom. These can be probed by a left detector (not shown) which measures operators indicated by solid (green) arrows and a right detector which measures operators indicated by dashed (orange) arrows. Arrows are labelled by the corresponding spin algebra operators, such as $\sigma_{x,L} \equiv -i\gamma_3\gamma_5$ in (a). The four possible bipartitionings are depicted with the corresponding measurement operators (a)–(d). CHSH correlations are necessarily violated in at least one of the four partitionings. Therefore any allowed state in the topologically protected ground space is non-locally entangled.

Quantum systems generically realise both entangled and unentangled states. The novel aspect of Romito and Gefen's work is that they show that, in the degenerate space spanned by MZM, any state is nonlocally entangled. That is, (at least) one bipartitioning of the MZM can always be found which violates the CHSH inequality.

For their minimal complexity setup, a generic state of the four-dimensional odd parity ground space is

$$|\psi\rangle = Ad^\dagger_{1,3}d^\dagger_{4,2}d^\dagger_{5,6}|0\rangle + Bd^\dagger_{1,3}|0\rangle + Cd^\dagger_{4,2}|0\rangle - Dd^\dagger_{5,6}|0\rangle, \tag{2.44}$$

where $|A|^2 + |B|^2 + |C|^2 + |D|^2 = 1$ and the zero energy fermionic degrees of freedom are

$$d_{1,3} = (\gamma_1 + i\gamma_3)/2, \tag{2.45}$$
$$d_{4,2} = (\gamma_4 + i\gamma_2)/2, \tag{2.46}$$
$$d_{5,6} = (\gamma_5 + i\gamma_5)/2, \tag{2.47}$$

and the state $|0\rangle$ is defined by $d_{1,3}|0\rangle = d_{4,2}|0\rangle = d_{5,6}|0\rangle = 0$. Notice that $\sigma_{z,L}, \sigma_{x,L}, \sigma_{y,L}$ satisfy the Pauli spin–1/2 algebra. Similar operators can be found on the right set, so that the state $|\psi\rangle$ reads

$$|\psi\rangle = A|\uparrow_L\uparrow_R\rangle + B|\uparrow_L\downarrow_R\rangle + C|\downarrow_L\uparrow_R\rangle + D|\downarrow_L\downarrow_R\rangle, \tag{2.48}$$

where $|\uparrow_i\rangle, |\downarrow_i\rangle$ are the eigenstates of $\sigma_{z,i}$, $(i = L, R)$. The maximal value of the CHSH correlation $\mathcal{C}$ is given by

$$\mathcal{C}_{135|246} = 2\sqrt{1 + 4|AD - BC|^2}, \tag{2.49}$$

where the subscript indicates the partitioning in which the measurement is performed. Maximal CHSH violations for the other three partitions are given by

$$\mathcal{C}_{564|132} = 2\sqrt{1 + |AC - DB|^2}, \tag{2.50}$$
$$\mathcal{C}_{421|563} = 2\sqrt{1 + |AB - CD|^2}, \tag{2.51}$$
$$\mathcal{C}_{641|352} = 2\sqrt{1 + 4|A^2 + C^2 + D^2 + B^2|^2}. \tag{2.52}$$

The pillar of Romito and Gefen's paper is that the condition $\mathcal{C}_{135|246} = \mathcal{C}_{564|132} = \mathcal{C}_{421|563} = \mathcal{C}_{641|352} = 2$ can never be satisfied. Spatially separated measurements of the system violate CHSH correlations in at least one of the four partitions. Hence any allowed state in the topologically protected ground subspace is nonlocally entangled.

This pervasive non-local entanglement occurs due to the topology of the system. It is interesting to study quantum correlations not necessarily due to entanglement. For this reason, we compute quantum discord in a setup similar to Romito and Gefen.

Chapter 3

Model and computational methods

We are interested in studying quantum discord in the simplest six Majorana zero mode model. In order to compute the discord, we will need a full model describing the bulk, not only the Majorana zero energy end-states. This is provided by the Kitaev chain.

3.1 Kitaev chain

The Kitaev chain is a one-dimensional model proposed by Kitaev [1] in which Majorana zero modes (MZM) –a type of localised quasiparticle –emerge at the edge points. The model consists of a tight-binding chain of spinless electrons and a superconducting term. Physical realisations of Kitaev chains have been engineered experimentally with semiconductor wires [6, 7, 8] and magnetic impurity chains [39]. Kitaev's Hamiltonian reads

$$H_{\text{Kitaev}} = \sum_{j=1}^{N} -\mu\Big(a_j^\dagger a_j - \frac{1}{2}\Big) + \Delta a_j a_{j+1} + \Delta^* a_{j+1}^\dagger a_j^\dagger - w\Big(a_j^\dagger a_{j+1} + a_{j+1}^\dagger a_j\Big). \tag{3.1}$$

The terms correspond to the chemical potential energy μ, the Cooper pairing (i.e. superconducting) amplitude Δ and the nearest-neighbour hopping amplitude w, respectively. The j label the sites $1, \dots, N$ and the $a_j^\dagger$ and a_j are the Dirac fermion creation and annihilation operators, respectively. We rewrite them in terms of the Majorana fermion degrees of freedom, $\gamma_1, \dots, \gamma_{2N}$,

$$\gamma_{2j-1} \equiv a_j + a_j^\dagger,\ \gamma_{2j} \equiv \frac{a_j - a_j^\dagger}{i},\ j = 1, ..., N, \tag{3.2}$$

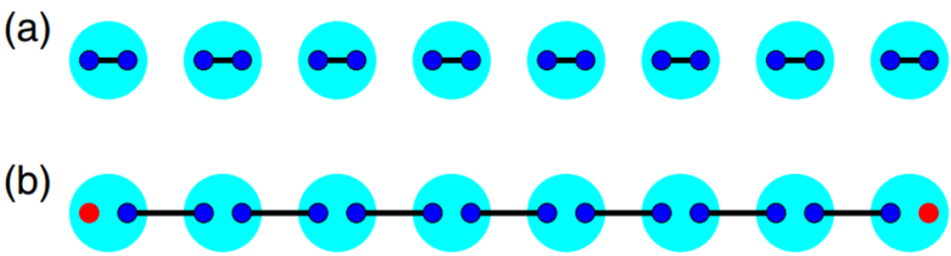


Figure 3.1: From S. R. Elliot and M. Franz [38]. Two phases of the Kitaev chain. (a) The trivial phase, $2|w| < |\mu|$: Majorana fermions on each lattice site can be thought of as bound into ordinary fermions. (b) The topological phase, $2|w| > |\mu|$: Majoranas on neighboring sites are bound, leaving two unpaired Majorana fermions at the ends of the chain.

which obey the Clifford algebra $\{\gamma_l, \gamma_m\} = 2\delta_{l,m}, \quad \gamma_l^\dagger = \gamma_l, \quad l, m = 1, 2, ..., 2N$. The Kitaev chain then reads

$$H_{\text{Kitaev}} = \frac{i}{2}\sum_j^{L-1}\Big[-\mu\gamma_{2j-1}\gamma_{2j} + (w + |\Delta|)\gamma_{2j}\gamma_{2j+1} + (-w + |\Delta|)\gamma_{2j-1}\gamma_{2j+2}\Big]. \tag{3.3}$$

We consider two special cases.

(a) The trivial case: $|\Delta| = w = 0, \mu < 0$

$$H_{\text{trivial}} = -\mu\sum_{j=1}^{L-1}\left(a_j^\dagger a_j - \frac{1}{2}\right) = \frac{i}{2}(-\mu)\sum_j^{L-1}\gamma_{2j-1}\gamma_{2j}. \tag{3.4}$$

The Majorana operators from the same site are paired together to form a ground state with occupation number 0.

(b) The topological case: $|\Delta| = w > 0, \mu = 0$

$$H_{\text{topological}} = iw\sum_j^{L-1}\gamma_{2j}\gamma_{2j+1}. \tag{3.5}$$

Here the Majorana operators from *different sites* are paired together. At the end points we find localised zero-energy MZM γ_1, γ_{2N} and the ground state shows a two-fold degeneracy, corresponding to even and odd parity subspaces.

3.1.1 A two-site Kitaev chain

An understanding of entanglement in Kitaev chains will be important in the analysis of the minimal complexity setup. We consider the simplest Kitaev chain, which consists of two fermion sites. The sites interact with a hopping term w and an induced superconducting gap Δ. We set the chemical potential to zero for simplicity. The situation is sketched in Fig. 3.2a. In order to compute the entanglement between the partitions, we require a full model describing the bulk, not only the Majorana zero energy end-states. Therefore the Hamiltonian is

$$H_{j=2} = -w(a_1^\dagger a_2 + a_2^\dagger a_1) + \Delta a_1 a_2 + \Delta^* a_2^\dagger a_1^\dagger, \tag{3.6}$$

Following Kitaev [1], we define Majorana fermions in the following way

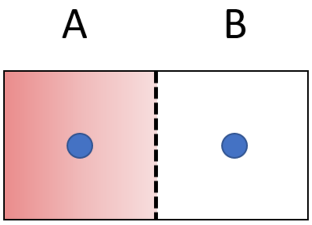
$$\gamma_1 \equiv a_1 + a_1^\dagger, \tag{3.7}$$
$$\gamma_1' \equiv (a_1 - a_1^\dagger)/i, \tag{3.8}$$
$$\gamma_2 \equiv a_2 + a_2^\dagger, \tag{3.9}$$
$$\gamma_2' \equiv (a_2 - a_2^\dagger)/i. \tag{3.10}$$

When $w = \Delta$, the chain is said to be topological and the superconducting coherence length is zero. The Hamiltonian may then be rewritten as

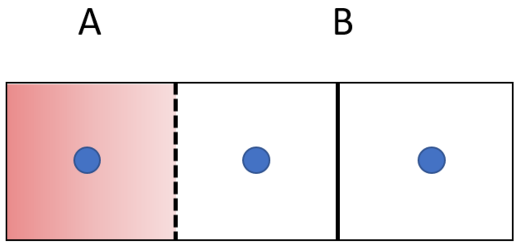
$$H_{j=2} = iw(\gamma_1'\gamma_2). \tag{3.11}$$

The finite energy states are $\frac{1}{2}(\gamma_2 + i\gamma_3)$ and $\frac{1}{2}(\gamma_2 - i\gamma_3)$. γ_1, γ_2' are zero energy eigenstates as they do not appear in the Hamiltonian. We pair them into Dirac fermions in the following way

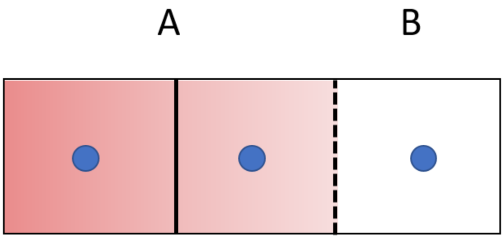
$$d = \frac{1}{2}(\gamma_2 + i\gamma_3), \tag{3.12}$$
$$d^\dagger = \frac{1}{2}(\gamma_2 - i\gamma_3), \tag{3.13}$$
$$d_0 = \frac{1}{2}(\gamma_1 + i\gamma_4), \tag{3.14}$$
$$d_0^\dagger = \frac{1}{2}(\gamma_1 - i\gamma_4). \tag{3.15}$$



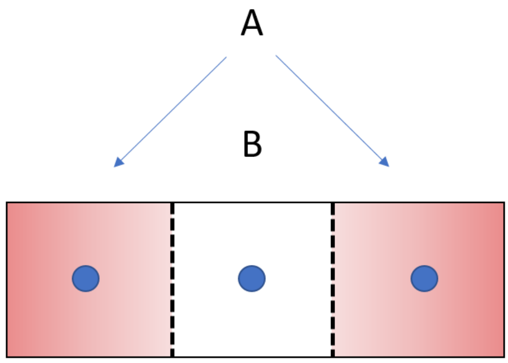
(a) Two-site Kitaev chain.



(b) Three-site Kitaev chain, composed of two subsystems: site one and sites two and three.



(c) Three-site Kitaev chain partitioned into sites one and two and site three.



(d) Three-site Kitaev chain partitioned into sites one and three and site two.

Figure 3.2: Blue sites are Dirac fermion degrees of freedom which interact via nearest neighbour hopping and Cooper pairing (i.e. superconductivity) terms. In the topological phase, Majorana zero modes emerge at the edges of the chain. We can partition the chain into two subsystems (red and white) by tracing over degrees of freedom. No matter where we partition the chain or the number of sites in the chain, we find the entropy to be the same, suggesting entanglement is due entirely to the Majorana zero modes and is independent of the length of the chain.

We choose the orthogonal ground states to satisfy $d|\tilde{0}\rangle = 0$ and $d|\tilde{1}\rangle = |\tilde{0}\rangle$. The basis for the two-site system is then $\{\,|\tilde{0}\tilde{0}\rangle,\ |\tilde{0}\tilde{1}\rangle,\ |\tilde{1}\tilde{0}\rangle,\ |\tilde{1}\tilde{1}\rangle\,\}$, where d acts on the first entry of the Kronecker product and d_0 onto the second. Since the states associated with d don't appear in the Hamiltonian $H_{j=2}$, the general equation for the ground states are

$$|\Psi\rangle = \alpha|\tilde{0}\tilde{0}\rangle + \beta|\tilde{0}\tilde{1}\rangle, \tag{3.16}$$

where $\alpha, \beta \in \mathbb{C}$ are to be determined. We want to solve this in terms of the original Dirac fermion basis $\{\,|00\rangle,\ |01\rangle,\ |10\rangle,\ |11\rangle\,\}$, so that we may trace over their degrees of freedom in order to compute the entanglement entropy. Accordingly, we write the ground states as

$$\begin{aligned} |\Psi_0\rangle &= A|00\rangle + B|01\rangle + C|10\rangle + D|11\rangle & (3.17)\\ &= A|00\rangle + Ba_2^\dagger|00\rangle + Ca_1^\dagger|00\rangle + Da_1^\dagger a_2^\dagger|00\rangle, & (3.18)\\ |\Psi_1\rangle &= A'|00\rangle + B'a_2^\dagger|00\rangle + C'a_1^\dagger|00\rangle + D'a_1^\dagger a_2^\dagger|00\rangle, & (3.19) \end{aligned}$$

where $A, B, C, D \in \mathbb{C}$ are to be determined and $|\Psi_0\rangle \equiv |\tilde{0}\tilde{0}\rangle$ and $|\Psi_1\rangle \equiv |\tilde{0}\tilde{1}\rangle$. To determine the constants, we have four equalities

$$\begin{aligned} d|\Psi_0\rangle &= 0, & (3.20)\\ d_0|\Psi_0\rangle &= 0, & (3.21)\\ d|\Psi_1\rangle &= 0, & (3.22)\\ d_0|\Psi_1\rangle &= |\Psi_0\rangle, & (3.23) \end{aligned}$$

which together with normalisation yield

$$\begin{aligned} |\Psi_0\rangle &= \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle), & (3.24)\\ |\Psi_1\rangle &= \frac{1}{\sqrt{2}}(|01\rangle + |10\rangle), & (3.25) \end{aligned}$$

which are even and odd parity ground states, as we expect. We now form the most general ground state, which is the pure state

$$|\Psi\rangle = \alpha|\Psi_0\rangle + \beta|\Psi_1\rangle, \tag{3.26}$$

constrained by $|\alpha|^2 + |\beta|^2 = 1$. To obtain the entanglement entropy of the state, we write the state as its density matrix

$$\begin{aligned} \rho &= \frac{1}{2}\big(\alpha|00\rangle + \beta|01\rangle + \alpha|10\rangle + \beta|11\rangle\big)\big(\text{h.c.}\big) & (3.27)\\ &= \frac{1}{2}\begin{pmatrix} |\alpha|^2 & \alpha\beta^* & \alpha\beta^* & |\alpha|^2 \\ \alpha^*\beta & |\beta|^2 & |\beta|^2 & \alpha^*\beta \\ \alpha^*\beta & |\beta|^2 & |\beta|^2 & \alpha^*\beta \\ |\alpha|^2 & \alpha\beta^* & \alpha\beta^* & |\alpha|^2 \end{pmatrix}. & (3.28) \end{aligned}$$

The reduced density matrices are

$$\rho^L = \rho^R = \frac{1}{2}\begin{pmatrix} 1 & \alpha\beta^* + \alpha^*\beta \\ \alpha\beta^* + \alpha^*\beta & 1 \end{pmatrix}. \tag{3.29}$$

This is all we need to compute the entanglement entropy $S(\rho^L) = -\operatorname{Tr}(\rho^L \log_2 \rho^L)$. However, finding logarithms of matrices is in general difficult. In the special case of diagonal matrices, computations are more straightforward. To diagonalise matrices, we make use of the spectral theorem. Accordingly, we find must find the eigenvalues of the reduced density matrices, which are $\frac{1\pm x}{2}$, where $x = \alpha\beta^* + \alpha^*\beta$. The entanglement entropy is then

$$S(\rho^L) = S(\rho^R) = -\frac{1+x}{2}\log_2(1+x) - \frac{1-x}{2}\log_2(1-x) + 1. \tag{3.30}$$

3.1.2 A three-site Kitaev chain

To devise a minimal complexity setup we must first thoroughly understand the behaviour of entanglement across Kitaev chains. We want to determine whether entanglement entropy depends on the length of the Kitaev chain or the choice of bipartitioning. Accordingly, we study a Kitaev chain consisting of three fermion sites, as depicted in Fig.3.2b-3.2d). In doing so we leave four-dimensional Hilbert space and enter eight dimensions. Computations become difficult to perform by hand so we make use of a Mathematica script to move forward. The original Dirac fermion basis is now

$$\{\, |000\rangle,\, a_1|000\rangle,\, a_2|000\rangle,\, a_1a_2|000\rangle,\, a_3|000\rangle,\, a_1a_3|000\rangle,\, a_2a_3|000\rangle,\, a_1a_2a_3|000\rangle \,\}, \tag{3.31}$$

which for simplicity, we refer to as

$$\{\, |000\rangle,\, |001\rangle,\, |010\rangle,\, |011\rangle,\, |100\rangle,\, |101\rangle,\, |110\rangle,\, |111\rangle \,\}. \tag{3.32}$$

In order to perform algebraic computations with fermionic operators, we represent them using matrices in Mathematica. The matrices satisfy the anticommutative fermion algebra and are defined as

$$a_1^\dagger = \begin{pmatrix} 0&0&0&0&0&0&0&0\\ 1&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0\\ 0&0&1&0&0&0&0&0\\ 0&0&0&0&0&0&0&0\\ 0&0&0&0&1&0&0&0\\ 0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&1&0 \end{pmatrix}, \tag{3.33}$$

$$a_2^\dagger = \begin{pmatrix} 0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0\\ 1&0&0&0&0&0&0&0\\ 0&-1&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0\\ 0&0&0&0&1&0&0&0\\ 0&0&0&0&0&-1&0&0 \end{pmatrix}, \tag{3.34}$$

$$a_3^\dagger = \begin{pmatrix} 0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0\\ 1&0&0&0&0&0&0&0\\ 0&-1&0&0&0&0&0&0\\ 0&0&-1&0&0&0&0&0\\ 0&0&0&1&0&0&0&0 \end{pmatrix}. \tag{3.35}$$

In order to compute the entanglement entropy, we need a full description, including the bulk, not only the MZM at the edges. The Hamiltonian for a Kitaev chain with three sites is

$$H_{j=3} = -w(a_1^\dagger a_2 + a_2^\dagger a_3 + a_2^\dagger a_1 + a_3^\dagger a_2) + \Delta(a_1a_2 + a_2a_3) + \Delta^*(a_2^\dagger a_1^\dagger + a_3^\dagger a_2^\dagger), \tag{3.36}$$

where w is the hopping amplitude and Δ is the induced superconducting gap. In the topological phase, $w = \Delta$, Majorana operators from different sites are paired together

$$H_{j=3} = iw(\gamma_1'\gamma_2 + \gamma_2'\gamma_3). \tag{3.37}$$

In this case the superconducting coherence length is zero. We can define new fermion annihilation operators

$$d_i := \frac{1}{2i}\Big(a_i - a_i^\dagger - a_{i+1} - a_{i+1}^\dagger\Big), \tag{3.38}$$

where $i \in \mathbb{Z}/3\mathbb{Z}$. Ground states satisfy the condition

$$d_1|\Psi_0\rangle = d_2|\Psi_0\rangle = d_3|\Psi_0\rangle = d_1|\Psi_1\rangle = d_2|\Psi_1\rangle = 0. \tag{3.39}$$

There are two orthogonal states $|\Psi_0\rangle$ and $|\Psi_1\rangle$ with this property. Indeed, Majorana oeprators at the edges remain unpaired, so can write

$$d_3|\Psi_1\rangle = |\Psi_0\rangle. \tag{3.40}$$

Solving these conditions, we obtain

$$|\Psi_0\rangle = \frac{1}{2i}\Big(|000\rangle + |011\rangle + |101\rangle + |110\rangle\Big), \tag{3.41}$$

$$|\Psi_1\rangle = \frac{1}{2}\Big(|001\rangle + |010\rangle + |100\rangle + |111\rangle\Big). \tag{3.42}$$

A general ground state is then

$$|\Psi\rangle \equiv \alpha|\Psi_0\rangle + \beta|\Psi_1\rangle. \tag{3.43}$$

The entropy of entanglement is defined as the entropy of either bipartitioning with the degrees of freedom of the other traced out. We trace out qubits one and two using M. Tame's code [40], obtaining ρ^{12}, and trace out qubits two and three to obtain ρ^{23}. Similarly, we can trace out qubits one and three to obtain ρ^{13}. The eigenvalues for all three density matrices are equal with values

$$\frac{1}{2}\left(1-\sqrt{-\left(\beta\alpha^*-\alpha\beta^*\right)^2}\right), \quad \frac{1}{2}\left(1+\sqrt{-\left(\beta\alpha^*-\alpha\beta^*\right)^2}\right). \tag{3.44}$$

We can trace out qubits $1, 2$ or 3 obtaining ρ^1, ρ^2, ρ^3, respectively. The eigenvalues for the density matrices are equal with values

$$0, \quad 0, \quad \frac{1}{2}\left(1-\sqrt{-\left(\beta\alpha^*-\alpha\beta^*\right)^2}\right), \quad \frac{1}{2}\left(1+\sqrt{-\left(\beta\alpha^*-\alpha\beta^*\right)^2}\right). \tag{3.45}$$

The entropy of entanglement is therefore

$$\begin{aligned} S(\rho^\lambda) = &-\frac{1-\sqrt{-\left(\beta\alpha^*-\alpha\beta^*\right)^2}}{2}\log_2\left(\frac{1-\sqrt{-\left(\beta\alpha^*-\alpha\beta^*\right)^2}}{2}\right) \\ &-\frac{1+\sqrt{-\left(\beta\alpha^*-\alpha\beta^*\right)^2}}{2}\log_2\left(\frac{1+\sqrt{-\left(\beta\alpha^*-\alpha\beta^*\right)^2}}{2}\right), \end{aligned} \tag{3.46}$$

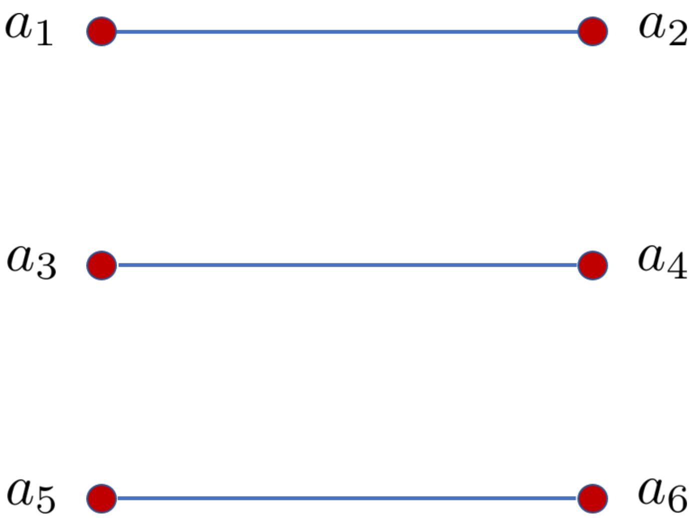
where λ labels a partial trace over either $1, 2, 3, 12, 23$ or 13. These results show that the entanglement entropy across a Kitaev chain is independent of the choice of partitioning. Further, the entropy for two sites is equal to that of three sites (up to a phase factor of -1), showing that the entanglement does not depend on the length of the chain, in the limit that the superconducting coherence length is zero.

3.2 Majorana-based device

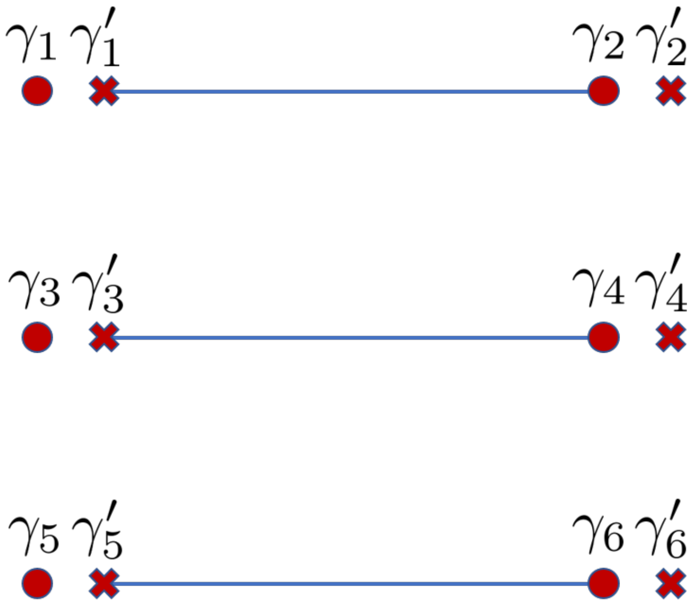
3.2.1 Minimal complexity setup

Simplest model

We are interested in studying quantum discord in a topologically protected system. The simplest model consists of six Majoranas. Sites one and two interact, as do three and four, and five and six. The interaction terms are a hopping amplitude w and a Cooper pairing (superconducting) amplitude Δ. For simplicity, we set the chemical potential to zero. In



(a) A simple model: six fermion sites form two partitions as in the most general model. In this case only fermions one and two, three and four, and five and six interact (with hopping term w and superconducting term Δ, shown in blue). In other words, the system consists of three two-site Kitaev chains which do not interact with each other.



(b) Topological case ($w = \Delta$): as in the most general model, MZM appear on the wire edges. The external detector measures unpaired Majorana occupancies. We find that states in the degenerate ground subspace obey the Pauli spin–1/2 algebra. Consequently, the quantum discord of Werner states is the same in both models.

other words, the system consists of three two-site Kitaev chains which do not interact with each other. The situation is depicted in figure 3.3a. In order to compute the discord, we need a full Hamiltonian describing the bulk, for which the Hamiltonian is

$$H_{\text{simple}} = -w\Big(a_1^\dagger a_2 + a_3^\dagger a_4 + a_5^\dagger a_6\Big) + \Delta\Big(a_1 a_2 + a_3 a_4 + a_5 a_6\Big) + h.c. \tag{3.47}$$

To compute the discord, we need the entanglement entropy of ground states in the partitioning. The Hilbert space for the partitioning is defined by Dirac fermion operators. We use matrices to represent their anticommutative algebra. In Mathematica, we write the $2^n \times 2^n$ sparse arrays

$$a_i^\dagger = \sum_i^{2^{i-1}} \text{SparseArray}\left[\left\{\text{Band}\left[\left\{2^{i-1}+i, i\right\}, \text{Automatic}, \left\{2^i, 2^{\mathrm{i}}\right\}\right] \to \prod_{j=1}^{i} (-1)^{\binom{i-1}{2j-1}}\right\}\right.$$
$$\left., \{2^n, 2^n\}\right], \tag{3.48}$$

where n is the dimension of the Hilbert space and states are taken to be in the standard computational basis. i labels the fermion operators. Sparse arrays are a computationally faster alternative to matrices if particular values appear frequently (in this case zero, one and negative one). We can create an off-diagonal sparse array using *Band*. *Automatic* fills in the array, starting at the element $(2^{i-1}+i, i)$ and moving across in steps of $(2^{i-1}, 2^{i-1})$, with the the value $\prod_{j=1}^{i}(-1)^{\binom{i-1}{2j-1}}$, where $\binom{i-1}{2j-1}$ is the binomial coefficient. The summation $\sum_i^{2^{i-1}}$ then adds the off-diagonal sparse arrays $1, \ldots, 2^{i-1}$ together, creating a banded sparse array.

We can define Majorana fermion degrees of freedom γ_i, γ_i' from the Dirac fermion degree of freedom a_i in the following way

$$\gamma_i = a_i + a_i^\dagger, \quad \gamma_i' = (a_i - a_i^\dagger)/i. \tag{3.49}$$

Majorana operators are required to obey the Clifford algebra

$$\{\gamma_\alpha, \gamma_\beta\} = 2\delta_{\alpha,\beta}, \quad \{\gamma_\alpha, \gamma_\beta'\} = 0. \tag{3.50}$$

The Kitaev wires in their topological phases ($w = \Delta$) are depicted in figure 3.3a. In this case, the superconducting coherence length is zero and the Hamiltonian becomes

$$H_{\text{general}} = iw\Big(\gamma_1'\gamma_2 + \gamma_3'\gamma_4 + \gamma_5'\gamma_6\Big). \tag{3.51}$$

The nonzero eigenstates are mixtures of the paired Majorana degrees of freedom

$$\gamma_6 + i\gamma_5', \tag{3.52}$$
$$\gamma_4 + i\gamma_3', \tag{3.53}$$
$$\gamma_2 + i\gamma_1', \tag{3.54}$$
$$\gamma_6 - i\gamma_5', \tag{3.55}$$
$$\gamma_4 - i\gamma_3', \tag{3.56}$$
$$\gamma_2 - i\gamma_1', \tag{3.57}$$

and the zero energy eigenstates are unpaired Majorana modes at the wire edges

$$\gamma_6', \gamma_5, \gamma_4', \gamma_3, \gamma_2', \gamma_1. \tag{3.58}$$

One can define new fermion annihilation operators

$$d_1 = -i\gamma_3' - i\gamma_5' - \gamma_2 + \gamma_6, \tag{3.59}$$
$$d_2 = i\gamma_1' + i\gamma_3' + \gamma_2 + \gamma_4, \tag{3.60}$$
$$d_3 = i\gamma_3' + i\gamma_5' - \gamma_2 + \gamma_6, \tag{3.61}$$
$$\tag{3.62}$$

which span Majorana sites $\gamma_1', \gamma_3', \gamma_5', \gamma_2, \gamma_4, \gamma_6$. From the unpaired Majorana modes, we can define operators

$$d_4 = \frac{\gamma_1 - i\gamma_2}{2}, \tag{3.63}$$
$$d_5 = \frac{\gamma_3 - i\gamma_4}{2}, \tag{3.64}$$
$$d_6 = \frac{\gamma_5 - i\gamma_6}{2}. \tag{3.65}$$

Ground states $|\Psi_i\rangle$ are defined by the condition

$$d_1|\Psi_i\rangle = d_2|\Psi_i\rangle = d_3|\Psi_i\rangle = 0. \tag{3.66}$$

There are eight orthogonal states which satisfy this property. We define $|\Psi_1\rangle$ by $d_i|\Psi_1\rangle = 0$, for $i = 1, \ldots 6$ and

$$|\Psi_2\rangle = d_5^\dagger d_4^\dagger|\Psi_0\rangle, \tag{3.67}$$
$$|\Psi_3\rangle = d_6^\dagger d_5^\dagger|\Psi_0\rangle, \tag{3.68}$$
$$|\Psi_4\rangle = d_6^\dagger d_4^\dagger|\Psi_0\rangle, \tag{3.69}$$
$$|\Psi_5\rangle = d_4^\dagger|\Psi_0\rangle, \tag{3.70}$$
$$|\Psi_6\rangle = d_5^\dagger|\Psi_0\rangle, \tag{3.71}$$
$$|\Psi_7\rangle = d_6^\dagger|\Psi_0\rangle, \tag{3.72}$$
$$|\Psi_8\rangle = d_6^\dagger d_5^\dagger d_4^\dagger|\Psi_0\rangle. \tag{3.73}$$

The Majorana subspace does not accommodate a well-defined number of fermions: it may exchange pairs of fermions with the underlying superconductor. Nevertheless, the parity of the Majorana system is a good quantum number and gives rise to two degenerate subspaces, each of a definite parity. States $|\Psi_1\rangle, \ldots, |\Psi_4\rangle$ belong to the even parity subspace and $|\Psi_5\rangle, \ldots, |\Psi_8\rangle$ belong to the odd subspace. Without loss of generality, we restrict ourselves to the four-degenerate even parity manifold.

Define a paritioning of the system by two sets: $\gamma_1, \gamma_3, \gamma_5$ belonging to the left (L) set and $\gamma_2', \gamma_4', \gamma_6'$ to the right (R) set. Romito and Gefen [10] describe a measurement procedure in which a coupled detector measures any combination of pairs of Majorana products from the left or right set

$$\sigma_{x,L} = -i\gamma_3\gamma_5,\ \sigma_{y,L} = -i\gamma_5\gamma_1,\ \sigma_{z,L} = -i\gamma_1\gamma_3, \tag{3.74}$$
$$\sigma_{x,R} = -i\gamma_6'\gamma_2',\ \sigma_{y,R} = i\gamma_2'\gamma_6',\ \sigma_{z,R} = -i\gamma_4'\gamma_2'. \tag{3.75}$$

Physically, this is a measurement of the occupancy of certain Dirac fermion degrees of freedom, constructed from the Majorana degrees of freedom. These operators satisfy the Pauli spin–1/2 algebra. Using Mathematica to proceed though the lengthy algebra, we obtain their eigenstates to be

$$|\uparrow_L\uparrow_R\rangle \equiv \frac{|\Psi_1\rangle + i|\Psi_2\rangle}{\sqrt{2}}, \quad |\uparrow_L\downarrow_R\rangle \equiv \frac{|\Psi_3\rangle + i|\Psi_4\rangle}{\sqrt{2}}, \tag{3.76}$$
$$|\downarrow_L\uparrow_R\rangle \equiv \frac{|\Psi_3\rangle - i|\Psi_4\rangle}{\sqrt{2}}, \quad |\downarrow_L\downarrow_R\rangle \equiv \frac{|\Psi_1\rangle - i|\Psi_2\rangle}{\sqrt{2}}. \tag{3.77}$$

The coupled detector can measure any operator of the form

$$\hat{O}_L = \cos\theta_L \sigma_{z,L} + \sin\theta_L \cos\phi_L \sigma_{x,L} + \sin\theta_L \sin\phi_L \sigma_{y,L}. \tag{3.78}$$

Romito and Gefen choose this observable as its expectation values are bounded, $-1 \leq \langle \hat{O}_L \rangle \leq 1$. They can then detect violations of the Bell [32], or Clauser-Horne-Shimony-Holt (CHSH) [33] inequalities, providing evidence of genuine quantum correlations. For our purposes, the spin–1/2 algebra allows us to compare the Majorana model with two-qubit systems. Furthermore, it enables us to measure non-classical correlations using Romito and Gefen's physically realisable detector.

More general model

We now present a more general model, in which the degenerate subspace still obeys the Pauli spin–1/2 algebra. As before, consider six qubits labelled in the same way. Sites from either partition may interact with any site in the other via a hopping term w and a superconducting term Δ. For simplicity, we again set the chemical potential to zero. The situation is depicted in figure 3.4a. The Hamiltonian is

$$\begin{aligned} H_{\text{general}} = -w\Big(& a_1^\dagger a_2 + a_1^\dagger a_4 + a_1^\dagger a_6 + a_3^\dagger a_2 + a_3^\dagger a_4 + a_3^\dagger a_6 + a_5^\dagger a_2 + a_5^\dagger a_4 + a_5^\dagger a_6\Big) \\ + \Delta\Big(& a_1 a_2 + a_1 a_4 + a_1 a_6 + a_3 a_2 + a_3 a_4 + a_3 a_6 + a_5 a_2 + a_5 a_4 + a_5 a_6\Big) \\ &+ h.c. \end{aligned} \tag{3.79}$$

In the Majorana basis with $w = \Delta$, the superconducting coherence length is zero and the Hamiltonian is

$$H_{\text{general}} = iw\Big(\gamma_1'\gamma_2 + \gamma_1'\gamma_4 + \gamma_1'\gamma_6 + \gamma_3'\gamma_2 + \gamma_3'\gamma_4 + \gamma_3'\gamma_6 + \gamma_5'\gamma_2 + \gamma_5'\gamma_4 + \gamma_5'\gamma_6\Big). \tag{3.80}$$

Now the Majorana operators from different sites are paired together, as depicted in figure 3.4b. Unfortunately, the rank of this Hamiltonian is two so that the ground-state space is too large to progress. We therefore flip the interaction energy of sites three and six, five and four, and five and six (depicted as green lines in figure 3.4a). The Hamiltonian is then

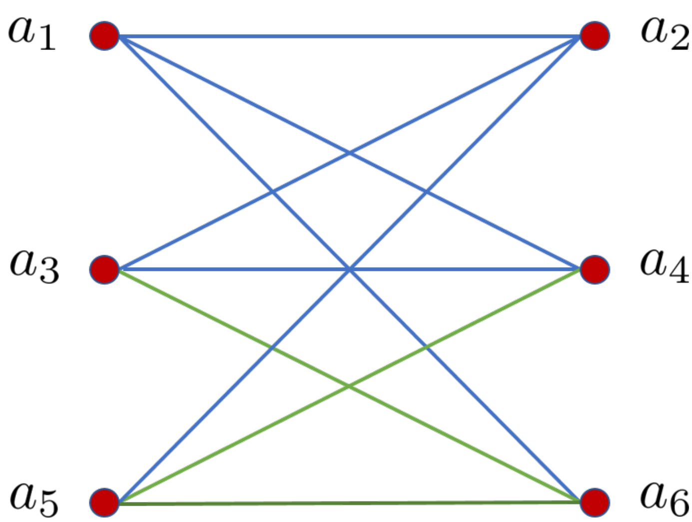
$$\begin{aligned} H_{\text{general}} = -w\Big(& a_1^\dagger a_2 + a_1^\dagger a_4 + a_1^\dagger a_6 + a_3^\dagger a_2 + a_3^\dagger a_4 - a_3^\dagger a_6 + a_5^\dagger a_2 - a_5^\dagger a_4 - a_5^\dagger a_6\Big) \\ + \Delta\Big(& a_1 a_2 + a_1 a_4 + a_1 a_6 + a_3 a_2 + a_3 a_4 - a_3 a_6 + a_5 a_2 - a_5 a_4 - a_5 a_6\Big) \\ &+ h.c. \\ = iw\Big(&\gamma_1'\gamma_2 + \gamma_1'\gamma_4 + \gamma_1'\gamma_6 + \gamma_3'\gamma_2 + \gamma_3'\gamma_4 - \gamma_3'\gamma_6 + \gamma_5'\gamma_2 - \gamma_5'\gamma_4 - \gamma_5'\gamma_6\Big). \end{aligned} \tag{3.81}$$

The eigenstates at finite eigenenergies are again mixtures of the paired Majorana degrees of freedom

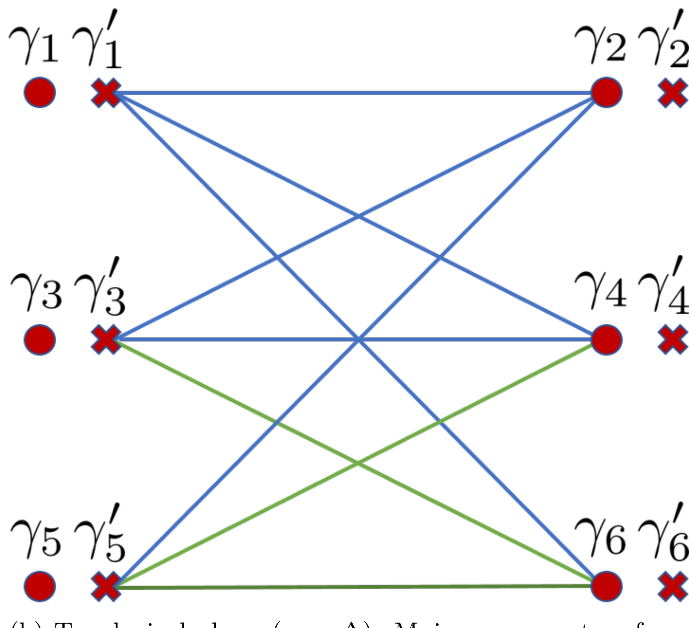
$$-i\gamma_3' - i\gamma_5' - \gamma_2 + \gamma_6, \tag{3.82}$$
$$i\gamma_1' + i\gamma_3' + \gamma_2 + \gamma_4, \tag{3.83}$$
$$i\gamma_3' + i\gamma_5' - \gamma_2 + \gamma_6, \tag{3.84}$$
$$-i\gamma_1' - i\gamma_3' + \gamma_2 + \gamma_4, \tag{3.85}$$
$$i\gamma_1' - i\gamma_3' + i\gamma_5' + \gamma_2 - \gamma_4 + \gamma_6, \tag{3.86}$$
$$-i\gamma_1' + i\gamma_3' - i\gamma_5' + \gamma_2 - \gamma_4 + \gamma_6. \tag{3.87}$$

The zero energy eigenstates are exactly those of the simple model

$$\gamma_6',\ \gamma_5,\ \gamma_4',\ \gamma_3,\ \gamma_2',\ \gamma_1. \tag{3.88}$$



(a) Minimal complexity setup: six fermion sites are partitioned into two sets: the left (L) set consists of fermions a_1, a_3, a_5 and the right (R) set consists of fermions a_2, a_4, a_6. Fermions from either partition may interact with any fermions in the other though a hopping term w and a superconducting term Δ, as shown in blue. Interaction terms between sites three and six, five and four, and five and six are required to have the sign of w and Δ reversed as shown in green.



(b) Topological phase ($w = \Delta$): Majorana operators from different sites pair together, leaving six unpaired zero energy Majorana operators. Pairs of these MZM can be probed by an external detector coupled to each partition. Non-classical correlations can then be measured. We quantum correlations which do not necessarily include entanglement.

One define fermion operators

$$d_1 = -i\gamma_3' - i\gamma_5' - \gamma_2 + \gamma_6, \tag{3.89}$$

$$d_2 = i\gamma_1' + i\gamma_3' + \gamma_2 + \gamma_4, \tag{3.90}$$

$$d_3 = i\gamma_1' - i\gamma_3' + i\gamma_5' + \gamma_2 - \gamma_4 + \gamma_6, \tag{3.91}$$

$$d_4 = \frac{\gamma_1 - i\gamma_2'}{2}, \tag{3.92}$$

$$d_5 = \frac{\gamma_3 - i\gamma_4'}{2}, \tag{3.93}$$

$$d_6 = \frac{\gamma_5 - i\gamma_6'}{2} \tag{3.94}$$

Ground states $|\Psi_i\rangle$ are defined in terms of d_i in exactly the same way as for the simple model, as is the coupled detector. Accordingly, we define $|\uparrow_L\uparrow_R\rangle$, $|\uparrow_L\downarrow_R\rangle$, $|\downarrow_L\uparrow_R\rangle$, $|\downarrow_L\downarrow_R\rangle$, in the exact same way as above and the spin–1/2 algebra will of course hold in this degenerate even subspace.

3.3 Calculation of quantum discord

To identify whether quantum correlations exist in the system which are not necessarily due to entanglement, we investigate the quantum discord which quantifies the difference between total and classical correlations (Section 2.1.5).

We consider the maximally mixed state

$$\rho_0 = \frac{|\uparrow_L\uparrow_R\rangle\langle\uparrow_L\uparrow_R| + |\uparrow_L\downarrow_R\rangle\langle\uparrow_L\downarrow_R| + |\downarrow_L\uparrow_R\rangle\langle\downarrow_L\uparrow_R| + |\downarrow_L\downarrow_R\rangle\langle\downarrow_L\downarrow_R|}{4}, \tag{3.95}$$

and a maximally entangled (singlet) state

$$\rho_- = \Big(\frac{|\uparrow_L\downarrow_R\rangle - |\downarrow_L\uparrow_R\rangle}{\sqrt{2}}\Big)\Big(\frac{\langle\uparrow_L\downarrow_R| - \langle\downarrow_L\uparrow_R|}{\sqrt{2}}\Big). \tag{3.96}$$

We introduce a parameter $c \in [0, 1]$ which takes us continuously from a maximally mixed state to the singlet state. This can be interpreted as the removal of noise from the system and is modelled by the *Werner state*

$$\rho \equiv (1 - c)\rho_0 + c\rho_-. \tag{3.97}$$

3.3.1 Total mutual information

To calculate the total mutual information we need two quantities: the entropy of the state of the entire system and the entropy of one of its two partitions. Accordingly, the nonzero eigenvalues of the Werner state are $\frac{1-c}{4}, \frac{1-c}{4}, \frac{1-c}{4}, \frac{1+3c}{4}$ so that the entropy is

$$S(\rho) = -\frac{3(1-c)}{4}\log_2(1-c) - \frac{1+3c}{4}\log_2(1+3c) + 2. \tag{3.98}$$

We define the state ρ^R by tracing over the degrees of freedom on the left (that is, tracing out the parity of branches 1, 3 and 5), which we perform using M. Tame's code [40]. Similarly, ρ^L is the state of the system with the right partition traced out. Of course, the entropies $S(\rho^L) = S(\rho^R)$. The eight-fold degenerate eigenvalues are 1/8, so that the entanglement entropy is

$$S(\rho^L) = 3. \tag{3.99}$$

From our results concerning Kitaev chains of two and three sites, we expect the entanglement entropy to be independent of the system length, in the limit that the superconducting coherence length is zero. For finite superconducting length, we expect small deviations from the value of the entropy. We have obtained the same result for the entropy of a system in which the junction was simple and in which the junction was more general. Accordingly, we expect the entanglement entropy for all six MZM systems to be independent of the particular configuration of the junction.

3.3.2 Classical mutual information

Measurements to obtain the classical mutual information are required to be of von Neumann type [12], that is that they must sum to the identity. Accordingly, for the L set $\gamma_1, \gamma_3, \gamma_5$ the coupled detector can measure any operator of the form

$$B_k = (I + k\hat{O}_L)/2, \tag{3.100}$$

where I is the identity operator on the state ρ, $k \in \{-1, 1\}$ parametrise the measurements and $\hat{O}_L$ is defined by Eq. 3.78.

If we perform a measurement $\{B_k\}$ locally on the L-set, then the quantum state, conditioned on the measurement outcome labelled by k, changes to

$$\rho_k = \frac{1}{p_k} B_k \rho B_k \tag{3.101}$$

with probability $p_k = \mathrm{Tr}(B_k \rho B_k)$. The nonzero eigenvalues of ρ_k are $(1-c)/2$, $(1+c)/2$. With this conditional density operator, an alternative variant of quantum conditional entropy (with respect to the measurement $\{B_k\}$) is defined as the average entropy

$$\begin{aligned} S(\rho|\{B_k\}) &= \sum_k p_k S(\rho_k) & (3.102) \\ &= -\frac{1-c}{2}\log_2(1-c) - \frac{1+c}{2}\log_2(1+c) + 1, & (3.103) \end{aligned}$$

and turns out to be independent of the specific measurement B_k and hence $\sup_{\{B_k\}} S(\rho|\{B_k\})$ $= S(\rho|\{B_k\})$. The classical mutual information is then

$$C(\rho) = \sup_{\{B_k\}} I(\rho|\{B_k\}) = S(\rho^L) - S(\rho|\{B_k\}). \tag{3.104}$$

3.3.3 Quantum discord

Now, we have two quantum analogues of the classical mutual information: the total mutual information, which are the mutual correlations of the subsystems; and the classical mutual information, which is the maximum information we can learn about one subsystem by performing measurements on the other. Their difference is interpreted as a measure of quantum correlations by Olliver and Zurek [12] and Henderson and Vedral [13] and is known as the quantum discord

$$\begin{aligned} Q(\rho) &= I(\rho) - C(\rho) & (3.105) \\ &= S(\rho^L) + s(\rho) + \sup_{\{B_k\}} S(\rho|\{B_k\}) & (3.106) \\ &= S(\rho^L) - 1 + \frac{1-c}{4}\log_2(1-c) \\ &\quad - \frac{1+c}{2}\log_2(1+c) + \frac{1+3c}{4}\log_2(1+3c). & (3.107) \end{aligned}$$

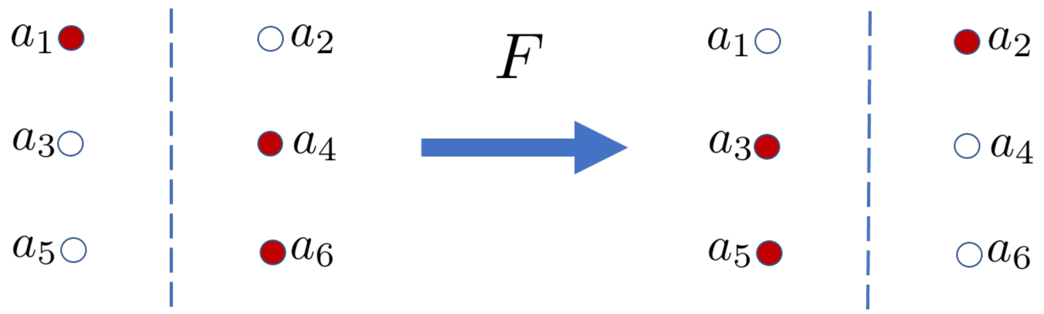


Figure 3.5: Flip operation in our setup. Defining the flip operation is a necessary step in Vollbrecht and Werner's calculation of the entanglement of formation for Werner states in arbitrary dimensions. The flip operator F flips the occupancy of each fermion site in the $\mathcal{H}^3 \otimes \mathcal{H}^3$ bipartite space, where the fermion occupancy is the occupancy of the respective pair of edge Majorana zero modes. White and red dots are occupied and unoccupied fermion sites, blue lines are superconducting wires which coalesce at the centre to form the junction.

3.3.4 Entanglement of formation

In order to compare the quantum discord with entanglement, we would need to compare it with the entanglement of formation, which is customarily used as a measure of entanglement [29, 31]. The entanglement of formation is defined for arbitrary dimensional bipartite systems. It quantifies the number of pure singlets needed to create a state with no further transfer of quantum information. Due to the optimisations involved, the entanglement of formation is hard to obtain analytically. Vollbrecht and Werner [41] ingeniously find the quantum discord for some special symmetric states.

Consider a composite Hilbert space $\mathcal{H} = \mathcal{H}_1 \otimes \mathcal{H}_2$. Two states ρ, ρ' are regarded as "equally entangled" if they differ only by a choice of basis in $\mathcal{H}_1$ and $\mathcal{H}_2$ or, equivalently, if there are unitary operators U_i acting on $\mathcal{H}_i$ such that $\rho' = (U_1 \otimes U_2)\rho(U_1 \otimes U_2)^*$. If in this equation $\rho' = \rho$, we call $U = (U_1 \otimes U_2)$ a (local) symmetry of the entangled state ρ.

In particular, consider the Hilbert space $\mathcal{H} = \mathcal{H}_1 \otimes \mathcal{H}_1$ and the group of unitaries of the form $U \otimes U$, where U is a unitary on $\mathcal{H}_1$. Vollbrecht and Werner find that $U \otimes U$ invariant states –*Werner states* –are of the form $\rho = \alpha I + \beta F$, where $\alpha, \beta \in \mathbb{C}$ such that $\operatorname{Tr}\rho = 1$, I is the identity operator and F is a 'flip operator' defined in the basis $|i\rangle$ of $\mathcal{H}_1$ in the following way

$$F = \sum_{i,j} |i,j\rangle\langle j,i|. \tag{3.108}$$

The general equation of the entanglement of formation of a Werner state in arbitrary dimensions is then

$$E(\rho) = \begin{cases} 0, & f(\rho) \leqslant 0, \\ \mathcal{E}(f(\rho)), & f(\rho) \geqslant 0, \end{cases} \tag{3.109}$$

where $\mathcal{E}$ was defined in Section 2.2 and

$$f(\rho) = -\operatorname{Tr}(\rho F), \tag{3.110}$$

parametrises ρ and ranges from -1 to 1. Unfortunately the Werner state in our setup cannot be parametrised by the identity and flip operation in $\mathcal{H}^3 \otimes \mathcal{H}^3$. The degenerate ground subspace $\mathcal{H}^4$ isn't seperable, so a flip operator in this space isn't defined. Further work to find the entanglement of formation should be done so that it may be compared to the quantum discord.

Chapter 4

Results

4.1 Entanglement entropy in Kitaev chains

4.1.1 Two-sites

We investigate the entanglement of a Kitaev chain at zero energy. We begin with the simplest model: a chain with two Dirac fermion sites depicted in Fig. 3.2a. Two Majorana operators are defined per fermionic site. Two eigenstates of the Hamiltonian are nonzero and corresponded to Majorana fermions paired into Dirac fermions at finite energy. There also exist two zero energy solutions corresponding to Majorana fermions at the edges. Following Kitaev [1], we pair them to form two zero-energy "Majorana zero modes" (MZM). At zero energy there exists an even and an odd parity ground state. From a superposition of these two states, we form the density matrix to simplify further calculations. As expected for factorisable states, the entropy is $S(\rho) = 0$ –see Eq. 3.28. In the density matrix representation, we are able to trace over the degrees of freedom of one of the two original fermion sites. The entropy of this quantity is a measure of entanglement [20] and is known as the entropy of entanglement. It is equal to

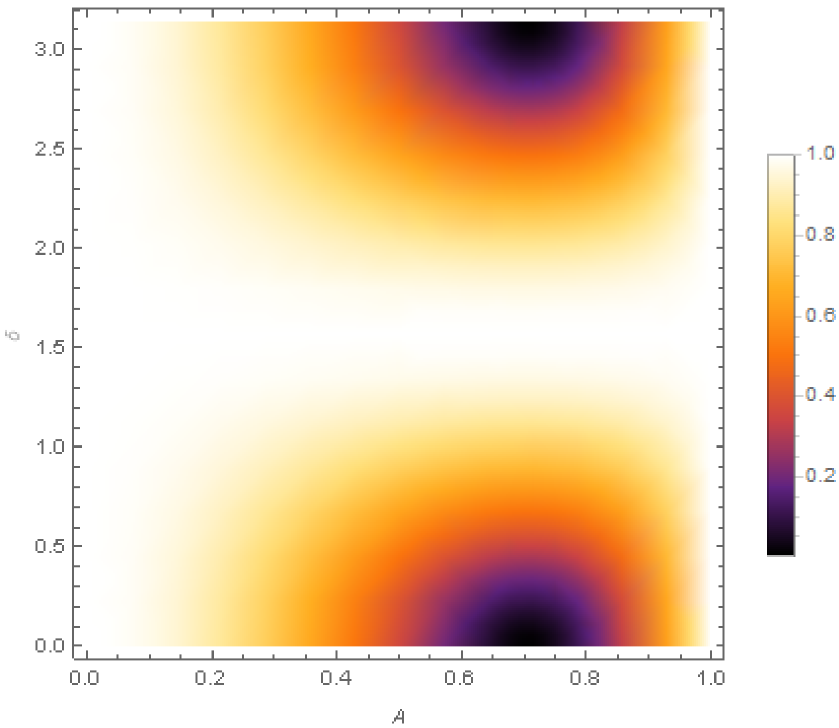
$$S(\rho^L) = S(\rho^R) = -\frac{1+x}{2}\log_2(1+x) - \frac{1-x}{2}\log_2(1-x) + 1, \tag{4.1}$$

where $x = \alpha\beta^* + \alpha^*\beta$. In our model, we hypothesis that it is a measure of entanglement between the zero energy states. To evaluate this hypothesis we will compare this result to the entropy of a three-site Kitaev chain.

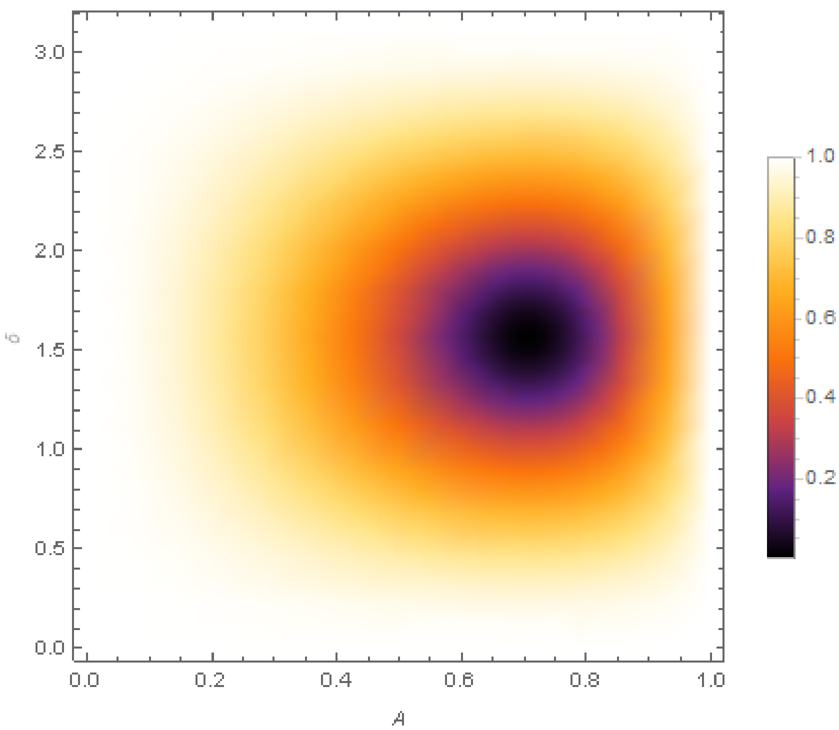
Now, observe that when the state is an equal superposition of even and odd states, $\alpha = \beta = 1/2$, the state exhibits no entanglement, $S(\rho^L) = 0$. When either the state is odd $\alpha = 0$, $\beta = 1$ or even $\alpha = 1$, $\beta = 0$, $S(\rho^L) = 1$ –maximal entanglement is achieved. Since $|\Psi\rangle$ is a superposition of these two instances, we expect a continuum of states for the entropies $S(\rho^L) \in [0, 1]$. Accordingly, we parametrise the amplitudes of the even and odd states by $\alpha \equiv Ae^{i\theta}, \beta \equiv Be^{i\phi}$, where A, B, θ, ϕ are real numbers. The phase difference between the odd even state $|\Psi_0\rangle$ and odd state $|\Psi_1\rangle$ is $\delta \equiv \theta - \phi$, so that the entropy is a function of two real values only

$$x = 2A\sqrt{1 - A^2}\cos\delta, \tag{4.2}$$

where $0 \leq A \leq 1$ is the magnitude of the amplitude of the even state and $0 \leq \delta < \pi$ is the phase difference between even and odd states. We plot the entropy in Fig. 4.1a as a function of A and δ. Here we see that the entropy is bounded by 0 and 1, as expected for a two-dimensional system (see discussion at the end of Section 2.1.3) and varies smoothly as we hypothesised above.



(a) Entropy $S(\rho^L)$ for the two-site Kitaev chain as a function of the amplitude of the even ground-state $-1 \leq A \leq 1$ and the phase difference $0 \leq \delta < \pi$ between the amplitudes of the two ground states. The entropy is bound between zero and $\log_2 \eta$, where $\eta = 2$ is the dimension of the Hilbert space.



(b) Similar plot to (a) for the three-site Kitaev chain. The results are the same up to a phase difference of π.

4.1.2 Three-sites

We seek to investigate the entanglement between bipartitions of the Kitaev chain in its topological phase. We expect that the entanglement will be due to existence of MZM. Therefore we expect to obtain the same entropy of entanglement regardless of where we perform the partition.

As above, there exist two zero-energy eigenstates –one odd, one even. We take a superposition of these states. Of course, it is pure so that $S(\rho) = 0$, as expected. Tracing over either side of a partitioning gives the same entropy of entanglement, $S(\rho^{23}) = S(\rho^1)$ (shown in Fig. 3.2b), $S(\rho^{12}) = S(\rho^3)$ (Fig. 3.2c) and $S(\rho^{13}) = S(\rho^2)$ (Fig. 3.2d), as we expect.

Interestingly, the six subsystem partitionings have equal entropies: let $0 \leq A \leq 1$ be the amplitude of the even parity eigenstate, on which the odd state depends on, and let $0 \leq \delta < \pi$ be their phase difference. Then the entropy of entanglement is

$$S(\rho^1) = -\frac{1+x'}{2}\log_2(1+x') - \frac{1-x'}{2}\log_2(1-x') + 1, \tag{4.3}$$

where $x' = 2A\sqrt{1-A^2}\sin\delta$. We plot this in Fig. 4.1b. This is the same result as obtained for a Kitaev chain with two sites, up to a phase difference π. Although we can always adjust phases, we cannot adjust the phase difference because the two ground states are are fixed by the constraints Eq. 3.39. We infer that the phase factor will alternate depending on the parity of fermions in the system.

Crucially, we have shown that no matter where we choose to bipartition the chain and regardless of the length of the chain, we find that the states associated with the MZM exhibit the same entropy of entanglement. We interpret this as the nonlocal entanglement of the MZM at the edges of the Kitaev chain.

4.2 Minimal complexity setup

4.2.1 Quantum discord

The minimal complexity setup (Fig. 3.4a) for our investigation consists of six fermions which form two partitions. Fermions from each partition interact with fermions in the other via a hopping term and an induced superconducing gap. Kitaev [1] has shown Majorana operators from different sites to pair when $w = \Delta$, leaving six unpaired MZM. Romito and Gefen [10] have devised a detection scheme to measure occupancies of the MZM. In our similar setup, we have reproduced their results that the four-fold degenerate subspace satisfies the Pauli spin–1/2 algebra. Furthermore, we have shown that the Pauli algebra holds in a simpler setup (Fig. 3.3a), involving fewer interactions.

The degenerate ground subspace is the four-dimensional Hilbert space $\mathcal{H}^4$. It is spanned by linear combinations of Dirac fermion occupancies in an $\mathcal{H}^3 \otimes \mathcal{H}^3$ Hilbert space. To perform calculations on the states, we change basis from the three fermionic degrees of freedom d_i which defined the ground states, to the Majorana γ_i degrees of freedom, which we can then rewrite in terms of the original six Dirac fermion sites a_i. This was a difficult step in the calculations and was performed using Mathematica (see for example, Eq. 3.48).

We then define the Werner state which is a linear combination of the maximally mixed and a maximally entangled state in the four-dimensional subspace $\mathcal{H}^4$. The Werner state depends on a parameter c. At $c = 0$, the Werner state is general: it is the maximally mixed ensemble of pure states in the degenerate ground subspace. As $c \to 1$, the Werner state traces out a path towards a maximally entangled state. Physically, the parameter may be interpreted as the removal of noise from the entangled state.

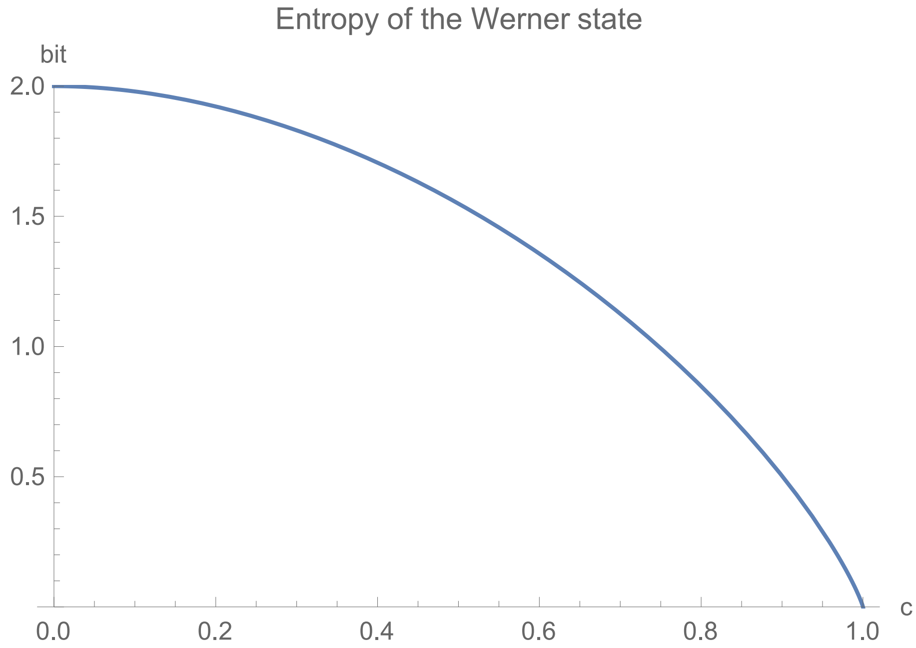


Figure 4.2: Entropy of the Werner state ρ as a function of its parametrisation $c \in [0, 1]$ in the Majorana-based device. $c = 0$ corresponds to the maximally mixed state in the four dimensional degenerate subspace. $c = 1$ is a maximally entangled pure state. Although the Werner state in the degenerate ground space exists in the inseparable four-dimensional space $\mathcal{H}^4$, the entropy is exactly the same as that of the two-qubit system –specified by the separable space $\mathcal{H}^2 \otimes \mathcal{H}^2$.

The Werner state has rank four and thus four eigenvalues. It obeys the same algebra as the states in two-qubit systems, however is not obvious whether inseparability of the Hilbert space will produce different eigenvalues from the two-qubit system. Interestingly, we find the eigenvalues are the same, hence obeying the Pauli algebra is sufficient to obtain the same entropy as the two-qubit state. We plot the entropy of the Werner state as a function of its parametrisation in Fig. 4.2. The entropy when the state is maximally entangled is zero, as expected for pure states. The maximally mixed state ($c = 0$) has entropy two, which corresponds to a maximally mixed state in the four dimensional degenerate subspace.

Tracing over the degrees of freedom of each partition yields the entropy of entanglement between the subsystems. Werner states in the Majorana setup and two-qubit systems are formed from states and operators which satisfy the Pauli algebra. Perhaps satisfying the Pauli algebra is enough for both systems to produce the same entropy of entanglement. Conversely, tracing over the larger Hilbert space of the Majorana setup may yield different eigenvalues than that of two-qubit systems.

We discover that the subsystem entropy is three, whereas recall for two-qubits it is one. This is the maximal entanglement in a space of three Dirac fermion degrees of freedom. It must be stressed that the von Neumann entropy of subsystems is a measure of entanglement only when states are separable, i.e. only when the Werner state is parametrised by $c = 1$. Unfortunately for mixed states, the von Neumann entropy is basis dependent and one must find the minimum over all basis for the state.

The entropy of the composite system and of its subsystems together measure the total (classical and quantum) correlations in the system. This quantity is known as the *total* *mutual information* and equals

$$I(\rho) = 2S(\rho^L) - S(\rho) \tag{4.4}$$

$$= \frac{3(1-c)}{4}\log_2(1-c) + \frac{1+3c}{4}\log_2(1+3c) + 4, \tag{4.5}$$

which is exactly the result Luo obtains for two-qubit systems, up to an additional constant of four. Physically, this constant signifies the existence of four additional bits of mutual information in the Majorana setup than that of two-qubit systems.

To compare the classical and quantum correlations, we now consider measurement induced correlations, which are interpreted as a measure of classical correlations by Olliver and Zurek [12] and Henderson and Vedral [13]. The MZM are partitioned into two separate sets, each of which is probed by an external detector. Physically, this is a measurement of the occupancy of the Dirac fermion degrees of freedom belonging to the partition. Measurements are separable for the two-qubit and Majorana setup. States in the two-qubit setup are also separable, that is they exist in the space $\mathcal{H}^2 \otimes \mathcal{H}^2$, whereas states in the Majorana setup exist in $\mathcal{H}^4$. Despite this, we find (using Mathematica to manipulate the lengthy algebra involved) that the quantum conditional entropy $S(\rho|\{B_k\})$ for the Majorana system is exactly the result obtained for two-qubits. Peculiarly, the quantum conditional entropy is independent of the measurement parameters and negates the difficult task of finding the supremum of the entropy.

The entropy of the state post-measurement is always less than or equal to that before the measurement. When the state is pure there is no uncertainty about the preparation of the state and the entropy is zero. Mixed states represent lack of certainty about the preparation of the state –it may have been prepared in one of several pure states. The entropy is a measure of the number of different possible pure states the mixed state was prepared from. Measurements reduce the uncertainty about the preparation and therefore reduce the entropy (see Fig. 4.3). Although the state after a measurement depends on the measurement parameters, its entropy doesn't. It appears that no matter which measurement we decide to perform, they all reveal the same amount of information about the preparation of the state.

From the (measurement-induced) quantum conditional entropy and the entanglement entropy of the subsystems we have the *classical mutual information*. It quantifies the maximum information we can obtain about the state of one subsystem by performing measurements on the other. In other words, it is a measure of the classical correlations which exist between the partitions. It is given by

$$C(\rho) = \sup_{\{B_k\}} I(\rho|\{B_k\}) \tag{4.6}$$

$$= S(\rho^L) - S(\rho|\{B_k\}) \tag{4.7}$$

$$= 2 + \frac{1-c}{2}\log_2(1-c) + \frac{1+c}{2}\log_2(1+c), \tag{4.8}$$

which is exactly the result for two-qubit systems, up to an additional constant of two. The Majorana system exhibits two additional classical bits of mutual information than that of two-qubit systems. This is as a result of the larger Hilbert space of the Majorana setup than that of two-qubit systems.

Now the difference between total and classical correlations can be interpreted as a measure of quantum correlations [12, 13]. This quantity is the *quantum discord*, with value

$$Q(\rho) = 2 + \frac{1-c}{4}\log_2(1-c) - \frac{1+c}{2}\log_2(1+c) + \frac{1+3c}{4}\log_2(1+3c). \tag{4.9}$$

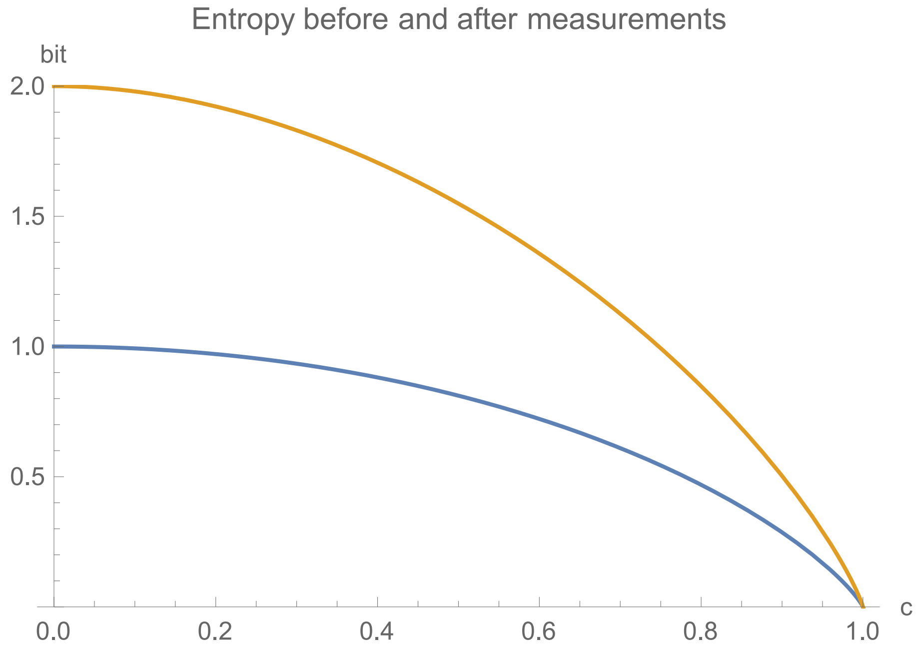


Figure 4.3: The entropy of the Werner state (gold) and the entropy post the measurement $\{B_k\}$ (blue) versus the parametrisation of the Werner state $c \in [0, 1]$. When the state is pure ($c = 1$), the entropy is zero and no further information may be extracted by performing a measurement. When the state is mixed ($c \neq 1$), measurements reveal more information about the state and hence reduce the entropy (or uncertainty) about the state.

We have obtained Luo's result for two-qubit systems (Fig. 2.2) with an additional constant of two. We plot the situation in Fig. 4.4 The larger Hilbert space of the Majorana setup than that of two qubits leads precisely to an additional two bits of classical and of quantum mutual information. Nevertheless, both quantities behave exactly as those of two-qubit systems.

We have shown that the quantum discord is ubiquitous for linear combinations of the maximally entangled state and a maximally mixed state in the degenerate ground subspace of a Majorana-based device. This is unsurprising as Romito and Gefen have shown that entanglement is ubiquitous for any state in this space. It is, however, non-trivial that the discord of the Majorana-based device, specified in $\mathcal{H}^4$ should behave exactly as that of two-qubit systems $\mathcal{H}^2 \otimes \mathcal{H}^2$.

4.2.2 Entanglement of formation

To determine whether nonclassical correlations in the Majorana-based device are due entirely to entanglement, we compare the quantum discord with the quantum entanglement. For pure states, the von Neumann entropy of either subsystem is the canonical measure of entanglement [20]. However for mixed states, the von Neumann entropy is basis dependent. For this reason we use a particular measure of entanglement, known as the entropy of formation [29], which minimises the entropy of a state over all its basis representations. This minimisation procedure makes the entropy of formation both a good measure of entanglement and a difficult one to compute. Analytical solutions have only been found in two-qubit systems [30, 31] and larger systems obeying particular symmetries [41]. For this reason, Luo [18] is able to compare his solution for the quantum discord of a two-qubit system with the entanglement of formation.

The Majorana setup consists of six qubits so the analysis is less straightforward. The

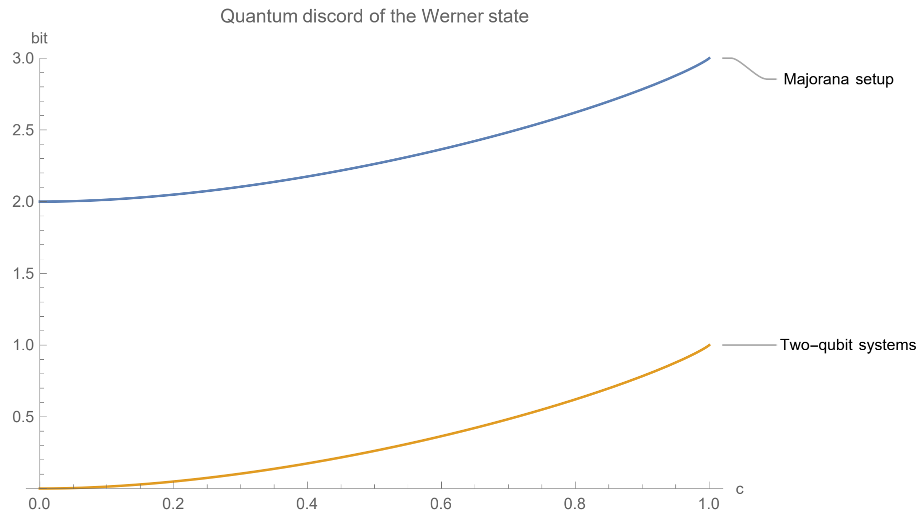


Figure 4.4: Quantum discord of the Werner state ρ as a function of its parametrisation $c \in [0, 1]$, which physically corresponds to the removal of noise. $c = 0$ corresponds to the maximally mixed state and $c = 1$ is the maximally entangled pure state. Although the Werner state in the Majorana setup exists in the inseparable four-dimensional space $\mathcal{H}^4$, the quantum discord is exactly the same as that of the two-qubit system –the separable space $\mathcal{H}^2 \otimes \mathcal{H}^2$, up to an additional constant of two which we attribute to the larger Hilbert space of the Majorana system.

Werner state in $\mathcal{H}^4$ is inseparable. Consequently, a flip operation is undefined. States in $\mathcal{H}^3 \otimes \mathcal{H}^3$ obey $U \otimes U$ symmetry. The Werner state however, cannot be parametrised by the identity and flip operation. Consequently, we have not found an analytical solution for the entanglement of formation. Further work to find a numerical computation should be done to advance the investigation. Another route to advance the investigation will be to find an alternative state in the degenerate ground subspace which satisfies $U \otimes U$ symmetries for which Vollbrecht and Werner have obtained analytical solutions.

Chapter 5

Conclusion

We have modelled the behaviour of entanglement in the ground state of a particular quantum wire, known as a Kitaev chain. The Kitaev chain consists of fermion sites aligned along one dimension. The sites may interact via a nearest-neighbour hopping term and a Cooper pairing term (an induced superconducting gap). A new operator basis is chosen, known as the Majorana basis, in which two Majorana operators are associated with each site. The chain exhibits a so-called *topological phase*, in which Majoranas from alternate sites pair up. Consequently, Majorana operators at the edges are left unpaired and exist at zero energy. We have quantified the entanglement of these "Majorana zero-modes" (MZM) using the von Neumann entropy and shown that it is independent of the choice of bipartitioning of the chain. Furthermore, the entanglement for a Kitaev chain consisting of two sites is equal to that of a chain consisting of three sites (up to a phase difference of -1), demonstrating entanglement to be independent of the length of the chain.

We have devised a minimal complexity Majorana-based device for investigating quantum discord. The device consists of six fermions which we partition into two sets. Fermions in each set interact with fermions in the other via a hopping term and a superconducting term. A Hamiltonian for the situation is obtained. In its topological phase, the system reveals six unpaired MZM. We find eight associated ground states. Due to the superconducting term, there exist two subspaces of even and odd parity. Without loss of generality, we restrict ourselves to the even degenerate subspace and show the states satisfy the Pauli spin–1/2 algebra. A less general model involving fewer interactions is also shown to obey the spin–1/2 algebra, suggesting that the algebra doesn't depend on the particular configuration of the junction.

To identify quantum correlations present in the system, we investigate the quantum discord for a Werner state. The Werner state traces out a path from the maximally mixed state in the degenerate ground subspace to a particular maximally entangled state. Total correlations, (measurement-induced) classical correlations and the quantum discord are all shown to be exactly equal to those of two-qubit systems, up to a constant. The constant represents ubiquitous additional information in the Majorana setup, not present in two-qubit systems. It is due to the larger Hilbert space of the Majorana setup. Measurements on one subsystem are shown to reveal equal bits of information about the other subsystem, regardless of the particular measurement performed.

Further work should be done to show the ubiquitousness of quantum correlations in the degenerate subspace. A straightforward approach will be to find the quantum discord for a path from the maximally mixed state to every maximally entangled state in the degenerate subspace.

An important problem is to determine whether quantum discord is due to entanglement or other quantum correlations. Few analytical solutions for the entanglement of formation exist. Therefore we propose two approaches: a numerical calculation of the entanglement of the Werner state or an analytical solution, taking advantage of the unitary $U \otimes U$ symmetry of the Majorana device. An identity and flip operation are then defined following Vollbrecht and Werner [41]. The operations generate a Werner state in $\mathcal{H}^3 \otimes \mathcal{H}^3$, which can then be projected into the degenerate ground subspace $\mathcal{H}^4$. Quantum discord can then be compared with the entanglement of formation.

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