Supplemental Material: Soft impurity walls and hard-wall branch promotion in a superconducting two-dimensional SSH model

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I. BDG CONVENTION

The normal-state Hamiltonian is the one given in Eq. (1) of the main text. All numerical calculations use the parameter convention

$$v = 0.6, \qquad w = 1.2, \qquad t_d = 0.9, \tag{1}$$

with periodic boundary conditions unless an explicit strip diagnostic is being plotted. For the imposed-pairing calculations the Nambu spinor is

$$\Psi = (c_A, c_B, c_A^\dagger, c_B^\dagger)^T, \tag{2}$$

and the local inter-sublattice pairing term is

$$H_\Delta(r) = \Delta_0 c_r^A c_r^B \Delta_0^* c_r^{B\dagger} c_r^{A\dagger}. \tag{3}$$

The reported fixed-pairing results use real positive $\Delta_0 = 0.3$. The global BdG eigenvector phase is arbitrary; component plots in the main text fix it by making the largest component real and positive.

II. WALL PROJECTOR AND SPECTRAL FUNCTION

The scalar wall is a one-cell-thick onsite potential support W_ℓ of length ℓ. The Hamiltonian itself contains only this support. For visualization and branch tracking, the wall-projected spectral weight is evaluated with a projector that includes a two-cell-wide strip around the wall and a two-cell pad along the wall direction. This finite projector prevents the diagnostic from becoming sensitive to a single lattice row, but it does not modify the BdG Hamiltonian.

For a fixed k_y, BdG eigenpairs (E_n, ψ_n) are computed and the positive-energy wall-projected spectral function is

$$A(k_y, E) = \sum_{n>0} P_{\text{wall}}^{(n)}(k_y) \frac{\eta_A/\pi}{[E - E_n(k_y)]^2 + \eta_A^2}, \qquad \eta_A = 0.08. \tag{4}$$

Here $P_{\text{wall}}^{(n)}$ is the norm of ψ_n on the wall projector. The plotted $A(k_y, E)$ is therefore an intensity diagnostic: its absolute scale depends on the projector and broadening, while the branch positions and spectral-weight redistribution are the robust quantities used in the paper.

III. HARD-WALL DESCENDANT BRANCH TRACKING

The fixed-pairing branch used in Figs. 6 and 7 of the main text is selected by hard-wall ancestry, not by taking the brightest ridge independently at each wall strength. The checkpoint calculation uses a 41×41 torus, $\ell = 41$, $\Delta_0 = 0.3$, $\mu = \delta\epsilon = 0$, 241 sampled k_y values for the plotted branch, and wall strengths

$$V = 0.1, 0.2, 0.5, 1.0, 1.5, 2.0, 2.5, 3.0, \ldots, 10, 12, 16, 32, 64, 100. \tag{5}$$

The operational branch-selection rule is:

1. At $V_{\max} = 100$, choose the positive-energy wall-supported state in the projected Weyl interval $|k_y|/2\pi \leq 1/3$.
2. Step downward in V.
3. At each step, choose the candidate eigenstate with the largest overlap with the previously selected eigenvector.
4. If nearby eigenvalues are nearly degenerate, compare the overlap with a nearby-state subspace rather than only a single vector.
5. Record ambiguity flags and nearby competitors in the checkpoint metadata.

The hard-wall-descendant checkpoint stores 'overlap_gap_tol = 0.04', 'degeneracy_window = 0.04', 'subspace_size = 4', and 'seed_energy_window = 0.8'. The same calculation records 66 ambiguous continuation entries over the full (V, k_y) grid. These flags are used as quality diagnostics; they are not interpreted as a phase boundary.

The normalized branch-promotion diagnostic in the main text uses

$$\eta_{\rm hw}(V) = \frac{|\epsilon_{\rm arc}(0;V) - \epsilon_{\rm arc}(0;V_{\min})|}{|\epsilon_{\rm arc}(0;V_{\max}) - \epsilon_{\rm arc}(0;V_{\min})|}, \qquad V_{\min} = 0.1, \quad V_{\max} = 100. \tag{6}$$

The largest jump in the 41×41 production run lies between $V = 2.5$ and $V = 3$. This is used as a finite-device onset window, not as a fitted thermodynamic critical point.

IV. SELF-CONSISTENT MEAN-FIELD ITERATION

The self-consistent calculations decouple an inter-sublattice attractive channel through

$$\kappa_{AB}(r) = \langle c_r^A c_r^B \rangle, \qquad \Delta_{AB}(r) = g_{AB}\kappa_{AB}(r). \tag{7}$$

The production wall-feedback checkpoints use

$$g_{AB} = 2.9249, \qquad \mu = 0, \qquad \delta\epsilon = 0.1, \tag{8}$$

on a 41×41 torus with a full wall. The fixed-point update is linearly mixed,

$$\Delta_{AB}^{(n+1)} = (1 - \eta_{\text{mix}})\Delta_{AB}^{(n)}\eta_{\text{mix}}\Delta_{AB}^{\text{new,(n)}}, \qquad \eta_{\text{mix}} = 0.35. \tag{9}$$

The checkpoint set reports the final residual after a fixed budget of 80 iterations. The residuals in the main-text table are therefore reproducibility diagnostics, not claims of exact fixed-point convergence to machine precision.

The spectral deformation atlas in the main text uses the interpolating field

$$\Delta_{AB}^{(\lambda)}(r) = (1 - \lambda)\Delta_0 + \lambda\Delta_{AB}^{\text{SCMFT}}(r; V), \qquad \lambda = 0, 0.5, 1. \tag{10}$$

Thus $\lambda = 0$ reproduces the fixed-Δ_0 reference at the same wall strength, while $\lambda = 1$ uses the relaxed SCMFT field.

V. FINITE-SIZE AND CONSISTENCY CHECKS

A thermodynamic-transition interpretation was tested and rejected using three checks. First, a branch-promotion diagnostic was extracted from the 41×41 tracked branch. Second, branch-aware emergence scans were compared across $L = 31, 41, 51, 61, 71$. Third, a hard-wall-character diagnostic was evaluated for

$$L = 21, 31, 41, 51, 61, 71, 81, 91, 101, 121, 141. \tag{11}$$

The stored finite-size diagnostic reports:

> window scan rejects critical exponent: beta spread=0.56, V_c spread=9.4; QCP not identified: require stable branch-character collapse and thermodynamic gap scaling.

The paper therefore uses the conservative language of finite-device branch promotion.

VI. CODE AND DATA PROVENANCE

The calculations and figure generation are implemented in the QuLab SSH research module [1]. The primary cached data files used for the publication figures are:

- `ssh_2d_fixed_delta_publication_atlas.npz`,
- `ssh_2d_weyl_arc_tracking.npz`,
- `ssh_2d_arc_transfer_qcp_assessment.npz`,
- `ssh_2d_scmft_boundary_stability.npz`,
- `scmft_interpolation_panels/ados/*.npz`.

The main text reports compact numerical slices of these data; the repository retains the full cached grids and omitted diagnostic figures.

[1] H. Sheehy, Qulab research module for two-dimensional ssh soft walls (2026), repository module: qulab.research.ssh_2d.